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SPECIAL ISSUE:

REFORMING THE MATHEMATICS CURRICULUM

A CHALLENGE FOR THE 90'S

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Improving Mathematics Instruction

by Robert Dixon
Douglas Carnine
University of Oregon

The articles in this issue deal with student achievement in mathematics: what is it, how did it get that way, and how can it be improved? The approach is based on *Theory of Instruction* (Engelmann and Carnine, 1982), particularly the analysis of quality samenesses. This analysis identified important samenesses and explicitly teaches them to students.

The contribution of a sameness analysis to teaching mathematics can be illustrated in geometry, where students learn equations, first for surface area and later for volume of various figures. Students are typically expected to learn seven formulas to calculate the volume of seven three-dimensional figures:

Rectangular prism: $l \cdot w \cdot h = v$
Wedge: $1/2 \cdot l \cdot w \cdot h = v$
Triangular pyramid: $1/6 \cdot l \cdot w \cdot h = v$
Cylinder: $\pi \cdot r^2 \cdot h = v$
Rectangular pyramid: $1/3 \cdot l \cdot w \cdot h = v$
Cone: $1/3 \cdot \pi \cdot r^2 \cdot h = v$
Sphere: $4/3 \cdot \pi \cdot r^3 = v$

These equations do not prompt higher-order thinking about volume, just the need to memorize formulas. The sameness analysis reduces the number of formulas students must learn from seven to slight variations of a single formula—area of the base times the height (B x h)—which brings conceptual coherence to exercises involving volume.

	Rectangular	
Prism	**Wedge**	**Cylinder**
$B \cdot h$	$B \cdot h$	$B \cdot h$
	Pyraimid	
Rectangular	**Triangular**	**Cone**
$B \cdot \frac{1}{3} \cdot h$	$B \cdot \frac{1}{3} \cdot h$	$B \cdot \frac{1}{3} \cdot h$
	Sphere	
	$B \cdot \frac{2}{3} \cdot h$	

For the regular figures—rectangular prism (box), wedge, cylinder—the volume is the area of the base times the height ($B \cdot h$). For figures that come to a point (pyramid with a rectangular base, pyramid with a triangular base, and a cone), the volume is not the area of the base times the height, but rather the area of the base times 1/3 of the height ($B \cdot \frac{1}{3} \cdot h$). The sphere is a special case—the area of the base times 2/3 of the height ($B \cdot \frac{2}{3} \cdot h$)—where the base is the area of a circle that passes through the center of the sphere, and the height is the diameter. The sameness analysis makes explicit the core concept that volume equals base times height. This core concept is obscured in math textbooks that present seven different formulas.

One of the primary purposes of the articles in this issue is to explore both the traditional mathematics curriculum and an alternative, *Connecting Math Concepts* (Engelmann and Carnine, 1991), that is built around important samenesses. Our contention is that once educators recognize the central role played by curricular material, they must demand empirically validated approaches that take into account not only the design of the curriculum, but the way the content is to be communicated and is to be implemented by teachers. In addition, these aspects (curriculum design, instructional delivery, and implementation) can and must be seen as being responsive to a full spectrum of students. In other words, teaching important samenesses can foster higher-order thinking in at-risk and learning-disabled students. However, designing curricular materials to accommodate lower-performing students does not have to "hold back" above-average students. The evaluation of *Connecting Math Concepts* indicates that higher performers are able to transfer what they've learned to solve very sophisticated problems. (The evaluation findings are summarized in the article by Carnine and Engelmann in this issue.)

Before examining math curricular material in more detail, it is important to understand the most current impetus for reforming mathematics education—the standards published by the *National Council of Teachers of Mathematics* (1989). Later in this introductory article, two aspects of these NCTM Standards will be discussed—the research base for the Standards and the historical context for the Standards. As is the case for most every educational reform, mathematics reform is hindered by insufficient data and a tendency to forget earlier reform efforts. The point of this issue is that earlier reforms failed, as the present one might, because of too little attention to pedagogy (e.g., the sameness analysis), the instructional delivery system, and implementation.

Overview of Remaining Articles

The next article in this issue, *Reforming the Mathematics Curriculum*, briefly describes some of the research on student performance in mathematics and on how mathematics instruction is conducted. The article then compares traditional basals and *Connecting Math Concepts* according to a series of criteria identified by Dixon (1990) in his review of the math and concept teaching research.

The article, *Making Connections in Third Grade Mathematics: Connecting Math Concepts*, looks more closely at how *Connecting Math Concepts* teaches important samenesses at a single grade level, across a variety of topics. The article, *Teaching Problem Solving in Mathematics*, examines how a single topic, word problems, can be taught across several grade levels.

The next article, *The Mathematics Curriculum—Standards, Textbooks, and Pedagogy: A Case Study of Fifth Grade Division*, analyzes the instruction in two math basals. The article also compares the versions of the basals that appeared prior to the publication of the NCTM *Standards* with the versions of the basals that were released after the *Standards*. This comparison indicates that the basic pedagogy of math textbooks has been largely untouched by the fervor over reform.

The last article, *Manipulatives—The Effective Way*, goes beyond the typical research that compares the use and nonuse of manipulatives. Rather, it investigates when to use manipulatives in the context of teaching regrouping. The results have implications for activities such as regrouping that come after a basic understanding of the relationship between manipulatives and numbers; it is more efficient to introduce the concepts and procedures first, and then present manipulative activities. Student understanding is as great as if manipulatives are introduced first, but far less instructional time is required.

This concludes the overview of the articles in this issue. The remainder of this article discusses the role of research and a historical perspective on reforming the mathematics curriculum.

The Role of Research in Mathematics Reform

The NCTM (1989) states three reasons for adopting and publishing the *Standards*:

1. "...to ensure that the public is protected from shoddy products."
2. "... as a means for expressing expectations about goals."
3. "...to lead a group toward some new desired goals" (p. 2).

Of these reasons, the first appears to apply most directly to research on mathematics education. With respect to that reason, NCTM asserts within the *Standards*: "It seems reasonable that anyone developing products for use in mathematics classrooms should document how the materials are related to current conceptions of what content is important to teach and should present evidence about their effectiveness" (p. 2). The *Standards* compare evidence of the effectiveness of mathematics programs with the kinds of evidence used by the Food and Drug Administration to establish minimum quality criteria for the distribution of drugs.

Given the NCTM's clearly stated desire for research-based *Standards* for effective mathematics instruction, the research base for the *Standards* appeared to us to be an excellent starting point for a review of mathematics research. However, we found the specific identification of the research base for the *Standards* to be illusive. In our attempts to identify that research base, we encountered a report by the NCTM's Research Advisory Committee (RAC) published prior to the publication of the *Standards* themselves (1988). Regarding this question of a research base for the *Standards*, the RAC report states:

> *The* Standards *document contains many recommendations, but in general it does not provide a research context for the recommendations, even when such a context is available (p. 339).*

and

> *Although there is no reason to expect a solid research base for every suggestion made in the document, the draft version did not distinguish those recommendations that were well-grounded empirically or theoretically from those that were based more on the informed judgment or personal opinions of the authors or that were drawn from examples and experience available in other countries (p. 339).*

These quotations speak for themselves: Some of the recommendations in the prepublication draft of the *Standards* were apparently based upon empirical or theoretical research, but that draft did not specify which. Such a rather vague reference to research does not seem commensurate with the goal of providing "evidence of effectiveness."

The RAC report (1988) anticipated that "The final version of the *Standards* document may clarify the basis for its recommendations more clearly, but it is likely that even more will need to be done" (p. 339). The final document, however, did not clarify the basis for its recommendations more clearly as pointed out by Bishop (1990) in his *Harvard Educational Review* article:

> *...it is a little surprising that there is not much reference to the research literature concerning mathematics learning and teaching. There is no impression of the existence of a substantial body of research on which, for example, the proposals in Standards are based. Recommendations and exhortations appear to be supported only by opinion—authoritative opinion, it is granted—but opinion nonetheless (p. 366).*

The Goals of Mathematics Education

Two of the three reasons given by the NCTM for establishing the *Standards* relate to mathematics education goals. While authoritative opinion does not provide the same kind of support for curricular standards as research does (Bishop, 1990), the authoritative opinion of mathematics educators is the principal basis for establishing goals of mathematics education.

The goals outlined in *Standards* for all students are that:

1. They learn to value mathematics.
2. They become confident in their ability to do mathematics.
3. They become mathematical problem solvers.
4. They learn to communicate mathematically.
5. They learn to reason mathematically.

Although the NCTM characterizes these laudable, but broad goals as "new," there is some evidence that they bear more than a slight similarity to broad goals of math education in the past. A brief review of mathematics education goals within this century helps frame the articles in this issue.

Rappaport (1976) identifies three "...distinct and significant periods..." in math education between 1900 and 1975. He characterizes the first, from 1900 to 1935, as the period of traditional mathematics, characterized by the "main aim" to "...teach children the skills that would enable them to solve problems of everyday life" (p. 566).

Rappaport identifies the second period, 1935-1958,

> *...as the period of meaningful arithmetic. The new aim was to have children understand the arithmetic concepts and the rationale behind the computational skills. There was an emphasis on the nature of the decimal numeration system as a place value system. Other numeration systems were presented as examples of place value systems. The emphasis was on arithmetic as a related system.* Problem solving was emphasized at all levels. *[emphasis added.] (p. 566).*

The content of the elementary school mathematics program during both periods described above, according to Rappaport, was basically the same. "The change was in the psychology of teaching and of learning rather than in the content."

Rappaport characterized the third period, from 1958 to the time of his writing as the era of new math, with an emphasis on changes in the content of the mathematics program, which he says school systems adopted with "uncritical rapidity." This change in content, rather than pedagogy, was emphasized by Macarow (1970):

> *One of the supposed strengths of the new math approach has been in the stress of self- experimentation, self-discovery and minimizing rote memorization while emphasizing the 'seeing' of mathematical structures which lie behind these systems. Following these pedagogical criteria in no way is to be identified with new mathematics: new mathematics is not to be interpreted as new and better ways of teaching (p. 396).*

Although new math switched to an axiomatic approach to mathematics, the broad goal of understanding remained from the earlier era of "meaningful arithmetic." Although the *goal* of the "meaningful arithmetic" era was clearly meaning, proponents of the new math charged that the goal hadn't been met.

> *A critical survey of mathematics textbooks completed during the last ten years has harvested a shocking number of useless definitions, downright errors, meaningless complications of simple concepts, emphasis laid on trivial aspects of a topic, and so on. (Wren, 1969, p. 443).*

Mathematicians defended the new math on the basis that it gave students a better understanding and appreciation of science and math (e.g., Wren, 1969), but critics like journalist Richard Martin (1973) responded, "There is one slight hitch: Many of these kids can't add, subtract, multiply, or divide."

Criticisms like this led to the movement that came to be called "back to basics." That movement, in turn, was criticized as a move away from understanding, as characterized in Offner's (1978) statement, "...the back-to-basics movement, which substitutes rote learning, 'consumer math,' and mindless pencil-pushing for understanding, is an educational crime" (p. 217).

Ironically, that criticism of back-to-basics is nearly identical to the criticisms that lead to new math. "The primary emphasis [of new math]," said Irving Cowle (1974), "is on insight and comprehension, *not* meaningless manipulation and reciting by rote. We want thinking, reasoning, and understanding, rather than mechanical responses to standard situations" (p. 71).

As Cooney (1988) has pointed out, the nature of the recent *Standards* developed in response to a lack of emphasis on understanding in the back-to-basics movement. Until recently, some mathematics educators have seemed to envision "understanding" and "adding, subtracting, etc." as mutually exclusive. Indeed, Rappaport concluded that only approximately 40% of students were capable of understanding mathematics. The math education for the remaining 60%, he recommended, should be limited to simple, practical arithmetic computation. At about the same time, however, Robert Davis (1974) suggested a more moderate course in which both computation and understanding could be accommodated, presumably for all students. "Today's math program should help children 'figure out the pattern' of a problem and then provide them with the skills to solve it correctly. Is this too much to ask? We think not" (p. 55).

From this discussion we see that:

1. Mathematics educators throughout the century have emphasized the establishment of broad goals within every period of mathematics education development this century.
2. In each period, perhaps with the expectation of back to basics, either problem-solving or understanding (or both) has been a central, highly emphasized goal of mathematics education.
3. Each new period of mathematics education has developed to some degree in response to the failure to obtain broad goals in the immediate preceding period. *Having* goals in each period never proved to ensure *achieving* goals.
4. In some cases, a period of mathematics education emphasized a major change in pedagogical approach (e.g., the discovery approach of meaningful math, 1935-1958) while another emphasized a major change in content (such as new math, with its emphasis on axiomatics).
5. Until recently, "mathematics understanding" was not necessarily a goal for all learners.

In the present series of articles we focus upon both pedagogical practice and approaches to content that appear to lead most reliably to the goals established by the NCTM in the *Standards*, which reflect the interest throughout most of this century on understanding *and* problem-solving. This focus is in keeping with the advice of Hill, Rouse, Wesson (1979), who asserted that "the responsible course [for mathematics education] is to identify sound principles of curriculum and instruction, whether they have their roots in the new math, in traditional arithmetic, or elsewhere."

In addition, we earnestly subscribe to the NCTM's aspiration that these goals be achieved by all learners—a complete rejection of Rappaport's (1976) suggestion that only 40% of students are capable of understanding mathematics. ◆

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Reforming Mathematics Instruction—*The Role of Curriculum Materials*

by Douglas Carnine,
University of Oregon

Educators are being caught between the rising expectations of reform and the changing composition of public schools. The need for reform in mathematics education has been made clear by the National Assesment of Educational Progress. Of U.S. 8th grade-age students, "only 16 percent of them have mastered the content of a typical 8th grade mathematics textbook; that is, they can (65 to 80 percent of the time) *'compute with decimals, fractions, and percents; recognize geometric figures; and solve simple equations.'* The vast majority of them, more than 2,800,000 out of 3,500,000, cannot do these kinds of tasks successfully at least 50 percent of the time" (Anrig & LaPointe, 1989, p. 7).

Yet the new standards from the National Council of Teachers of Mathematics (1989) don't even address the problems identified by the National Assessment, but go on to more far-reaching goals:

To value mathematics.
To reason mathematically.
To communicate mathematics.
To solve problems.
To develop confidence.

These are challenging goals even for students who historically have pursued math-related careers—white males. Changing demographics make these goals no less valuable, but far more difficult to reach. According to a report by the National Research Council, a private group that advised Congress on scientific issues (New York Times, April 11, 1990): "85% of new entrants into the work force are minorities and women, but few minorities and women enter engineering, science, and mathematics. Unless changes occur, the nation's needs for mathematically skilled teachers, scientists, engineers, and hosts of other workers for business, industry, and government will not be met. By the year 2000, the need for workers in these fields is expected to rise by 36% over the 1986 figure, the report said. But at the same time, demographic trends indicate that the traditional pool of scientists and engineers—white males—will fall at roughly the same rate. White males, presently the source of most elite workers in the mass production system, will constitute less than 10% of the net growth of our work force between now and 2000."

Forthcoming in The Journal of Behavior Education.

Additional impetus for reform comes from the relatively poor standing of U.S. students in international comparisons. Compared to Japanese students, almost 95% of our students are below average (International Association for the Evaluation of Educational Achievement, 1987).

In the United States, fraction instruction begins in grade one and repeats annually amidst numerous other mathematics objectives. In France, however, fractions are introduced and taught in a single grade (i.e., 7th grade). At the end of 7th grade, French students are more proficient in fractions than students in the United States (International Association for the Evaluation of Educational Achievement, 1987).

A final study reported that American 13-year-olds placed last in math and next-to-last in science when compared with students in four other countries and four Canadian provinces. Although U.S. students were last in mathematics knowledge, 58% said they were good at math. Conversely, although Korean students ranked highest in math, only 23% of that nation's students reported that they were good at math (LaPointe, Mead, & Phillips, 1989).

Problems in U. S. Mathematics Instruction

It seems the U.S. is well on its way to reaching the fifth broad goal of the National Council of Teachers of Mathematics—to instill confidence. However, developing competence will not be so easy, in part because of the ways in which mathematics instruction occurs and the structure of the textbooks that define the curriculum. The National Council of Teachers of Mathematics (1989) noted the need to change the "repetition of topics, approach, and level of presentation in grade after grade" (p. 66). This comment is directed at the spiral curriculum, in which each concept is revisited year after year. The intent of the spiral curriculum is to add depth each year, but the practical result is the rapid, superficial coverage of a large number of topics each year.

According to Porter (1989), a relatively large percentage of the topics taught in mathematics receive brief coverage. On the average, teachers devote less than 30 minutes in instructional time across the entire year to 70% of the topics they covered (e.g., telling time might receive 25 minutes during all of 1st grade). Teachers called this practice "teaching for exposure" and seemed comfortable with its use. Teaching for exposure has become commonplace in our classrooms, largely due to the fact that the practice parallels the recommendations for topic coverage in mathematics textbooks, which are trying to cover too many topics. Teachers are in a bind—they are expected to teach many, many topics, but most things take a lot of time to teach well. Research on two sequences for teaching "borrowing" (Evans and Carnine, this issue) found that the first sequence (manipulatives were introduced first followed by an algorithm), required an average of forty-one 10-minute, teacher-directed sessions. The other sequence (teaching the algorithm first, then manipulatives), took far less time—thirty-four 10-minute sessions on average. However, even if three sessions were scheduled each day, 11 to 14 days would still be required to teach borrowing with manipulatives and an algorithm, far more time than is allocated in math textbooks.

When individual differences are considered, the unreasonableness of the basal programs' expectations is even more apparent. When manipulatives were introduced first, some students required as much as 510 minutes of instruction, about seventeen 30-minute sessions.

Even 11 days of borrowing would be tedious for students and teachers. Undoubtedly, that is why basal math programs typically spend less than half as much time on borrowing. The trade-off is that many students will not have had enough instruction and practice to learn to borrow. When these students return to borrowing the next year in the spiral curriculum, they will receive even less instruction, leading to repeated failure and frustration. This downward cycle has been exacerbated by the trend to include more topics at each grade level, pushing fractions down into 1st grade or even kindergarten, for example.

Redesigning Math Instruction: Strands

There is an alternative. Rather than organizing an entire lesson around a single topic, as is done in traditional basal programs, lessons can be designed around strands; each 5- to 10-minute segment addresses a different topic. There are several reasons for organizing a curriculum around strands, utilizing shorter segments on various topics within each lesson.

First, students are more easily engaged with a variety of topics. For example, 30 minutes on borrowing day in and day out would become quite tedious. In contrast, a lesson consisting of 8 minutes on borrowing followed by 6 on estimation, 3 on facts, and 15 on word problems will be more likely to keep students engaged. Variety increases attentiveness and how much students learn. Working 30 or 40 problems consisting of a mix of borrowing, estimation, facts and word problems is reasonable; working 30 or 40 of just borrowing problems in a lesson is not.

Second, strands make the sequencing of component concepts more manageable. A mathematics curriculum contains many concepts. Arranging these concepts in a scope and sequence such that they are taught prior to their integration is possible only when several of them can appear in one lesson. For example, before students are introduced to borrowing, they learn to rewrite 37 as 20 + 17. This component is better taught in three 7-minute segments, spread over three days, rather than in a single 30-minute lesson. [For research on teaching component concepts before the more complex concepts, see Carnine (1980a) and Kameenui and Carnine (1986).]

Third, lessons composed of several segments make cumulative introduction feasible. In cumulative introduction, after a concept is introduced it is systematically reviewed and integrated with other related concepts. Cumulative introduction, as an alternative to the traditional spiral introduction, has three important advantages: (a) As noted earlier, components can be introduced early, (b) practice can be provided on both new and previously introduced concepts until responses are accurate and rapid, and (c) distributed practice on some concepts can occur every day. For example, only one or two difficult math facts would be introduced at one time. They would appear several times in every lesson for several consecutive lessons (massed practice). Once students became proficient at recalling those facts, the facts would be practiced less frequently in each lesson (distributed practice). Distributed practice is easy to schedule when each lesson is designed to accommodate several segments from several strands.

Organizing a curriculum by strands, in which several topics are covered in a lesson, is but one aspect of traditional math basals that must be reformed if the needs of a full spectrum of student abilities are to be met. Other criteria for reform have to do with *the use of time, the rate at which new concepts are introduced, the clarity and coherence of activities and explanations, and the adequacy and appropriateness of practice and review.*

One curriculum has been developed according to these criteria, with the intent of accommodating a wide range of student abilities—the mathematics curriculum used by the Direction Instruction Model.

This mathematics basal, *Connecting Math Concepts* (Engelmann and Carnine, 1991), stands in stark contrast to all traditional basals. The uniqueness of the curriculum is a major factor in the effectiveness of the Direction Instruction Model.

In a major national study (Stebbins, St. Pierre, Proper, Anderson, & Cerva, 1977), economically disadvantaged and handicapped students who participated in Direct Instruction in kindergarten through third grade performed as well as their more advantaged peers. In that study, the Direct Instruction system was also compared with other educational approaches (Gersten & Carnine, 1984). They ranged from Piagetian-derived approaches to open classroom models, psychodynamic approaches, and several models based on discovery learning. The testing in the schools and the data analyses were carried out by an independent research group. The third graders in over a dozen Direct Instruction school districts scored at the 48th percentile on the math section of the Metropolitan Achievement Test. The mean percentile for all the other approaches (except for that of the University of Kansas) was below the 20th percentile. A confirmation of these findings came from interviews with parents conducted by the Huron Institute (Haney, 1977). Parents of students in Direct Instruction felt their children were getting a better education than did parents of students in any other approach. Moreover, Direction Instruction students' scores were also highest on measures of self esteem, responsibility for success in school, and responsibility for failure in school.

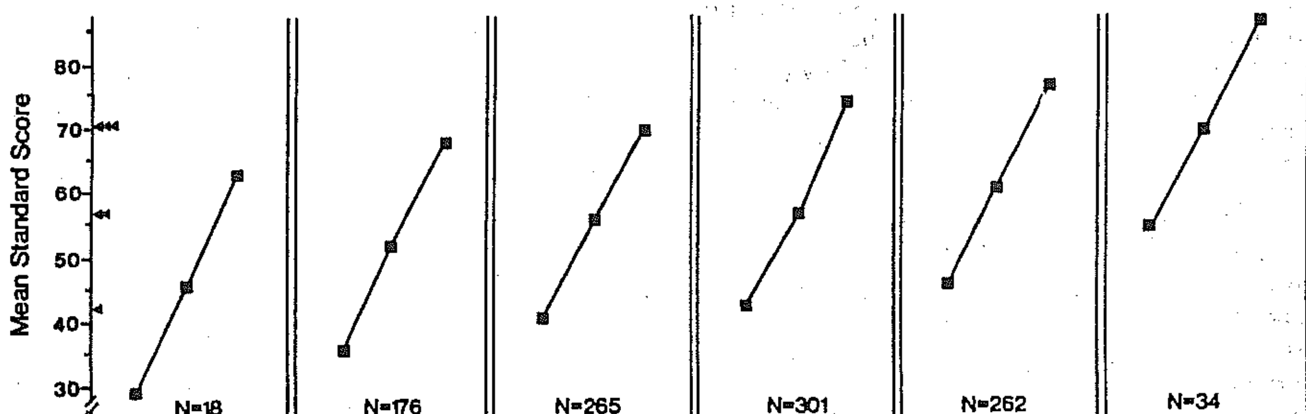
Out of the thousands of Direct Instruction students included in the study, 321 students were not economically disadvantaged. These students scored well above the third-grade level in mathematics (Gersten & Carnine, 1984):

4.3 in problem solving (the 75th percentile)
4.4 in concepts (the 68th percentile)
4.8 in computation (the 83rd percentile)

In a related finding, Gersten, Becker, Heiry, & White (1984) reported that while students entering Direct Instruction with relatively low IQ's scored lower on entry level mathematics tests than did students who entered with higher IQ's, both groups gained at least one grade-equivalent unit per year. (See Figure 1.) In addition, students who entered Direct Instruction with an IQ of over 111 did not, as a group, experience regression toward the mean, which would be expected. In other words, students entered kindergarten at different levels of understanding. At the end of third grade, student performance still differed substantially for students of differing ability. However, every ability group made significant progress each year.

Individual studies dealing with multiplication, division, fractions, ratios, proportions, and their associated word problems have also been conducted. In these studies the treatments included active teaching techniques but compared different curricula. All studies, which are summarized in Table 1, included low-performing students. In all studies, the effectiveness of Direct Instruction tended to be confirmed.

Figure 1. MAT Total Mathematics 1: Longitudinal Progress by IQ Block for Children in EK Sites (N=1.056)



IQ	N	Grade	Std. Score	S.D.	%ile	G.E.
70 and Below	18	1	29.2	7.3	10th	1.2
		2	46.1	10.1	10th	2.0
		3	62.6	9.5	24th	3.2
71 - 90	176	1	35.8	8.0	31st	1.5
		2	51.4	8.5	22nd	2.3
		3	67.7	10.6	39th	3.5
91 - 100	265	1	40.0	9.0	46th	1.7
		2	55.2	8.2	39th	2.5
		3	69.7	9.7	47th	3.7
101 - 110	301	1	42.6	9.2	51st	1.8
		2	56.9	8.8	50th	2.7
		3	73.5	10.7	61st	4.0
111 - 130	262	1	45.6	10.8	57th	2.0
		2	60.2	10.0	63rd	3.0
		3	76.4	11.6	69th	4.3
Above 130	34	1	54.4	10.6	83rd	2.5
		2	69.2	8.5	84th	3.6
		3	85.1	8.1	88th	5.3

◂ Grade 1 National Median
◂◂ Grade 2 National Median
◂◂◂ Grade 3 National Median

Other Important Design Features

As mentioned earlier, one critical aspect of *Connecting Math Concepts* is the organization of lessons around strands, not a single topic. The rest of this article illustrates the other aspects of *Connecting Math Concepts* that set it apart from traditional basals.

Use of Time

The students who require more teaching in school are usually the students who do not get help at home. They need even more instruction in school. Where will the time come from? The primary source of additional instructional time is in the math period itself. In *Connecting Math Concepts*, most of the period is devoted to interactive teaching, rather than the extensive, independent practice that occurs with basals. Another tactic is to completely drop topics that are intended to be "taught for exposure." In *Connecting Math Concepts*, the time saved by dropping inappropriate topics is devoted to high priority topics.

Even activities for topics that are a high priority must be designed to be efficient. The time allocated to math instruction must be used to maximize student learning. Efficiency concerns are greatest around the use of manipulatives, as noted in the earlier research on borrowing (Evans, 1990). Introducing manipulatives before the algorithm required an average of 90 extra minutes for each student. The inefficient use of manipulatives with a more advanced topic is illustrated in Figure 2. A paraphrase of a basal's suggestion for teaching two-digit divisor problems is given.

As Baroody (1989) noted, "...instruction should begin with experiences that are real to students..." (p. 4). By the time two-digit divisor problems are introduced, the concept of one-digit divisor problems should be "real." Thus, time-consuming manipulative activities for two-digit divisor problems may not necessary. For example, in problem A at the top of the page, the time required for a classroom of students to break 188 counters into units and then divide them into 31 groups could be spent more efficiently. Efficiency becomes even more important in reviewing the requirements for the rest of the page—students are to work nine more two-digit divisor problems with manipulatives.

Although manipulatives are essential for establishing basic number concepts and counting, other less time-consuming representations, such as pictures, can be used to teach the concepts of subtraction (Kameenui, Carnine, Darch, & Stein, 1986), multipli-

Table 1. Established Research Base in Mathematics

Authors	Topic	Results
Gleason, Carnine, & Boriero (in press)	Teaching multiplication and division word problems to middle-school students	Direct Instruction students, taught by a teacher or computer, progressed from a chance level to a 90% accuracy level.
Moore & Carnine (1989)	Teaching ratio and proportion word problems to secondary students.	Direct Instruction students had higher posttest scores than students receiving active teaching with an enhanced traditional curriculum.
Kelly, Gersten, & Carnine (1990)	Teaching fraction concepts to secondary students.	Direct Instruction students made fewer conceptual errors than students receiving traditional instruction.
Kelly, Carnine, Gersten, & Grossen (1986)	Teaching fractions to secondary students.	Direct Instruction students scored higher on post and maintenance tests than students receiving active teaching with an enhanced traditional curriculum.
Darch, Carnine, & Gersten (1984)	Teaching multiplication and division word problems to middle-school students	Direct Instruction students scored higher on post and maintenance tests than students receiving enhanced traditional curriculum and instruction.

Figure 2. Fifth Grade Basal: Introduction to Division by 2-digit Division to get a 1-digit Quotient.

Using the Pages To Teach Using Manipulatives

Refer students to Example A at the top of the page. Provide each group with a hundred-square, base ten blocks and unit squares.

After they have modeled 188 as one hundred, 8 tens and 8 ones, students should recognize that the hundred cannot be divided into 31 groups. They will have to rename the hundred as tens to have a total of 18 tens.

Then encourage the students to recognize that 18 tens cannot be divided into 31 groups, but rather they should rename the tens as ones for a total of 188 ones. They can then divide the ones into 31 groups with the same number of ones in each group. The ones that are left over constitute the remainder.

After each group has demonstrated 188 ÷ 31 with manipulatives, have them write the division example. The students should then use their manipulatives to work the other nine problems on the page.

One-Digit Quotients

A. Work in a group and explain your thinking as you proceed.

The School Advisory Committee has $188 to spend on awards for each classroom. There are 31 classrooms. So how much can they spend on each award? How much money will be left?

Use your place value materials to find 188 ÷ 31. Show 1 hundred, 8 tens, 8 ones. Rename in order to divide 188 into 31 equal groups. Explain your method to other students.

Try: Work in a group. Use place-value materials as you record your work.

a. $24\overline{)76}$ b. $11\overline{)80}$ c. $30\overline{)91}$ d. 153 ÷ 21

Practice: Work in a group. Use place-value materials as you record your work.

(1) $22\overline{)178}$ (2) $25\overline{)76}$ (3) $15\overline{)63}$ (4) $11\overline{)69}$ (5) $22\overline{)177}$

cation (Carnine, 1980a), division (Kameenui et al., 1986), fractions (Kameenui, Carnine, Darch, & Stein, 1986; Kelly, Carnine, Gersten, & Grossen, 1986), ratios (Moore & Carnine, 1989), and so forth. As Resnick and Omanson (1987) noted, "perhaps any discussion of quantities manipulated in written arithmetic, without any reference to the block analogue, could be just as successful in teaching the principles that underlie written [math] instruction" (p. 90). In *Connecting Math Concepts*, manipulatives are used with all new operations, *after* pictorial representations. The reason for beginning with pictorial representations is that time is saved and nothing is lost in conceptual understanding.

Rate for Introducing New Concepts

Because basal math texts typically devote an entire lesson to a single topic, they tend to introduce topics at a brisk rate—one a day. At times, this rate is too slow. Some topics do not deserve a full class period. The other problem is more troublesome—when a day isn't long enough for lower-performing students to learn and apply the new concepts subsumed by that day's topic. This problem is illustrated in Figure 3, which describes the seven fraction lessons found in a widely accepted third-grade math basal.

Note that each lesson represents a substantially new topic, at a pace far too fast for even average students. Moreover, no topic within a lesson is related to any previously introduced topic. In some lessons, such as 45, 46 and 47, the relationships between the topics aren't clear and are likely to confuse many students:

Lesson 45—a fraction is related to one whole. students represent a fraction by coloring a portion of a region.

Lesson 46—a fraction is related to a set. Given a set, students write the fraction that represents a subset.

Lesson 47—a fraction is related to a set. Given a fraction, students determine the subset.

Lesson 45 deals with a fraction as represented by

ReformING Mathematics Instruction—*Continued*

Figure 3. Rate of introduction of Fraction Topics in a Third Grade Basal

The key aspects of the first seven objectives in a third-grade basal are listed below. This material is introduced over 12 pages of text.

Objective 45: A fraction can be used to name a part of the whole. (Fold a rectangle into 2 equal parts, then 4. Determine how many different ways you can make the fold.) Color different parts—write the fraction.

Objective 46: A fraction can be used to name part of the set of pencils, markers, or crayons. (Number of red pencils, etc., over number of pencils, etc., in a box, e.g., 4 red crayons and 17 crayons in all. The fraction is 4/17.)

Objective 47: In 1/3, the 3 means 3 equal groups and the 1 means 1 of the groups. (Make 3 groups. Draw 3 flowers, put one in each group. Keep placing flowers in each group until you've placed a total of 15 flowers.)

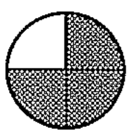
Objective 48: To find 1/3 of 12 mentally, divide 12 by 3. (You have 12 apples. You're going to cook 1/3 of them. How many apples will you cook? $12 \div 3 = 4$ 1/3 of 12 = __.)

Objective 49: Point out that numerators can be compared if the denominators are alike. (Compare two pizzas, each cut into 6ths. One has 5 pieces left over versus the other one with 3 pieces left over. Compare 5/6 and 3/6. 5/6 is greater than 3/6.)

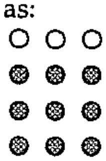
Objective 50: Use place value to explain that fractions with 10 as a denominator can also be written as decimals. Fraction, 1/10. Decimal, 0.1 = one-tenth. (Using hundreds square and ten sticks.)

Objective 51: Show that just as cents can follow dollars, so tenths and hundredths can follow whole numbers. (Using hundred square as the number 1, students write decimals, e.g., .001, 2.72, from pictures.)

a portion of a region, e.g., 3/4 is:



The concept of a fraction as a region is not related to the concept of a fraction as a subset, the concept covered in lessons 46 and 47. In lesson 46, students work from a set such as:

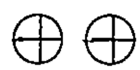


The students are to write the fraction

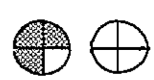
$$\frac{3}{4}$$

What is the relationship between a fraction as a region in lesson 45 and this new concept of a fraction in lesson 46? The basal is silent on this issue. In lesson 47, the student is given a fraction such as 3/4 and told to identify the subset of a set of 12 members. The progression of objectives for lessons 45 through 51 helps explain why most U.S. students can't make much sense out of fractions.

Connecting Math Concepts devotes all third grade instruction on fractions to that representation of fractions seen in lesson 45 of the basal, with one important difference—an emphasis on depth of understanding. First, students learn that fractions can represent values greater than one whole. The bottom number tells how many parts to divide each whole into; e.g., for 3/4 and for 7/4, students would divide the wholes in this fashion:

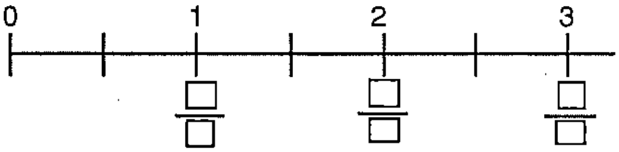


The top number tells how many parts the students have in the fraction; e.g., for the fraction 3/4, students shade three parts:



for the fraction 7/4, students shade seven parts:

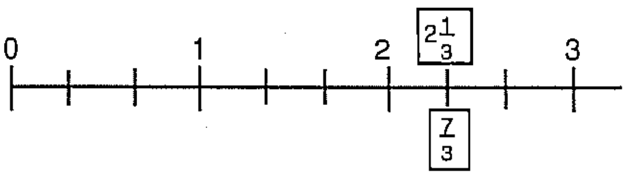
Students later learn whether fractions are less than one (2/3, 1/4), equal to one (7/7, 3/3), or greater than one (7/4, 4/3). The greater-than, less-than concepts prepare students to relate fractions to whole numbers on a number line, in exercises such as this:



Students write the following fractions in the boxes:

$$\frac{2}{2} \qquad \frac{4}{2} \qquad \frac{6}{2}$$

These exercises are important for students to understand that fractions and whole numbers are part of the same number system. Later, exercises relate mixed numbers and fractions:

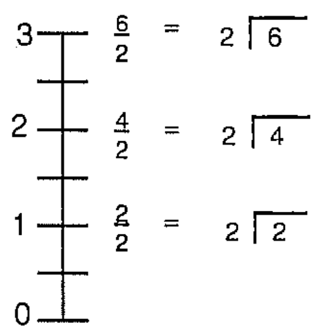


These exercises relate fractions and whole numbers to measurement, showing, for example, that 2-1/3 inch is the same as 7/3 inches.

Finally, students learn the relationship between fractions and division. Students learn to write 6/2 as:

$$2\overline{)6}$$

which is illustrated on a vertical number line:



At this point, the interrelationships between whole numbers, fractions and division are demonstrated. The goal of *Connecting Math Concepts* is to thoroughly teach the fundamental concept of a fraction and its relationship to other math concepts. The basal text, which teaches a different aspect of fractions every lesson and doesn't develop their interrelationships, is more likely to lead to student frustration and confusion.

Explanations and Activities

The explanations and activities in basal math textbooks usually have dual objectives—to develop conceptual understanding and procedural mastery. Basal textbooks typically rely on discovery; the students are to discover important concepts. Discovery does not work for many students who learn better from clear, explicit explanations of what to do and why to do it. The problems with discovery are most apparent with complex concepts, such as dividing fractions. A paraphrase of the introduction of dividing fractions from a widely-accepted basal appears in Figure 4 (suggestions to the teacher) and Figure 5 (the corresponding student page from the textbook).

Ironically, the goals of teaching for understanding and for procedural mastery work against each other. Many students are unlikely to discover the concept through these activities and also end up confused about the procedure itself. Moreover, the procedure is demonstrated in a rote fashion; in example 3 on the student page (Figure 5), why are the numbers crossed out and replaced by other numbers? Throughout the basal program, students are shown this as a rote procedure.

Figure 4. Basal Suggestions to the Teacher for Dividing Fractions

Objective: Divide a fraction or a mixed number by a fraction (first introduction of reciprocal).

Introduction: Using Manipulatives

Have students place their rulers on their desk. Rulers should be marked in at least eighths of an inch. When answering the following questions, have students count on their rulers. Then have them multiply by the reciprocal of the divisor.

a) How many 1/2 inches are there in 5 inches? $\boxed{10}$

b) How many 3/8 inches are there in 4 5/8 inches? $\boxed{15}$

c) How many 7/8 inches are there in 2 5/8 inches? $\boxed{3}$

Teach: Using the Pages

For Example 1, tell the students to find out how many groups of 5/8 are in 5 5/8 inches.

For Example 2, tell students that what they are finding out is how many groups of 3/8 there are in 3.

If the students use the ruler as a picture model, it will be easier to write the equation.

For Example 3, point out that multiplication and division are reciprocal operations, noting the reciprocal relationship the students derived in this example.

For Example 4, discuss how using a ruler gives the students a way to check their answer for reasonableness.

In *Connecting Math Concepts*, students build on their understanding of fractions, particularly upon the concept of fractions equal to one, such as 5/5, 7/7, or 39/39. With this conceptual understanding, students can comprehend the rationale for all the crossing out, as in example 3 of Figure 5:

$$\frac{45 \times 8}{8 \times 5}$$

By rewriting this expression as:

$$\frac{8 \times 45}{8 \times 5}$$

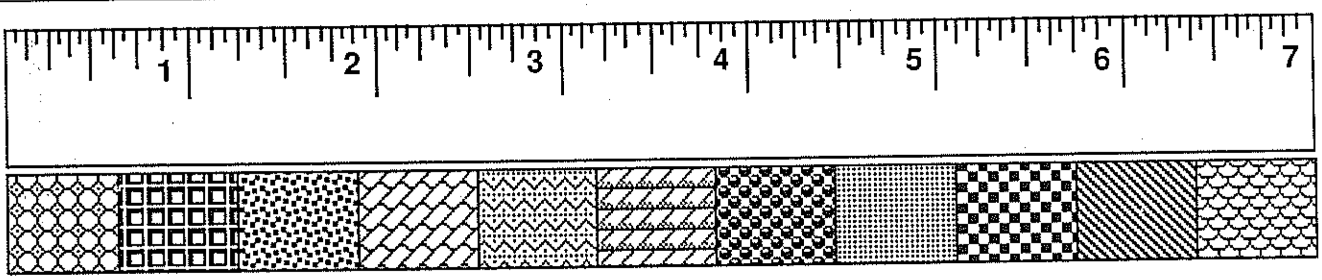
the fraction equal to one, eight over eight, is readily recognized. Because students have also been taught that any number times 1 equals the original number, students see that the equation is the same as:

$$1 \times \frac{45}{5}$$

which equals 45 over 5. Similarly, students in *Connecting Math Concepts* will have the background to see that

$$\frac{45}{5} = \frac{5 \times 9}{5 \times 1}$$

Figure 5. Student Basal Page for Dividing Fractions



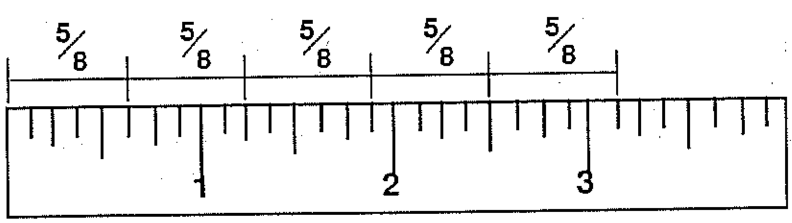
1. A strip of fabric pieces is shown below the ruler. Each piece of fabric on the strip is 5/8" long. To find the number of 5/8-inch pieces there are in 5-5/8 inches of fabric, you can count each piece. To your partner, explain how the equation 5 5/8 ÷ 5/8 = 9. Remember, to find how many 5/8 are in 5 5/8, you must divide 5 5/8 by 5/8. The quotient is 9.
2. Again, work with your partner. This time, show how to find the number of 3/8 in 3. Use the rule above and write the division equation using 3 and 3/8.
3. Study the following multiplication equations with your partner.

$$5\,5/8 \times 8/5 = 45/8 \times 8/5 = \frac{\cancelto{9}{45} \times \cancelto{1}{8}}{\cancelto{1}{8} \times \cancelto{1}{5}} = \frac{9}{1} = 9$$

$$3 \times \frac{8}{3} = \frac{3}{1} \times \frac{8}{3} = \frac{\cancelto{1}{3} \times 8}{1 \times \cancelto{1}{3}} = \frac{8}{1} = 8$$

How is 5/8 related to 8/5? How is 8/3 related to 3/8?

4. Discuss with your partner the multiplication problems above in Example 3 with the division problems in Examples 1 and 2. State the general rule for dividing by a fraction.
5. Work with your partner to figure out 3 1/8 ÷ 5/8 using the ruler. See the ruler. Do your answers agree?



Again, 5 over 5 equals one, so

$$\frac{5 \times 9}{5 \times 1} = 1 \times 9 \text{ or } 9.$$

Students who understand fractions don't cross out numbers in a rote fashion. They learn the lawful, reasonable nature of mathematics, in this case based on the identity element for multiplication: Any value multiplied by one, or a fraction equal to one, yields the original value.

Basal math programs almost universally revert to discovery for the introduction of difficult concepts. The only exception is solving word problems, for which almost nothing is offered, discovery or anything else. The suggestions for introducing the teaching of various types of word problems in a widely-accepted fifth 5th grade basal follow:

Addition and subtraction. These word problems are introduced very early in the program:

> "Ask students if each problem describes a joining situation, a removing situation or a comparing situation."

On another early lesson, the teacher is to "ask students if the answer will be greater or lesser than the greatest number in the problem."

Two pages later:

> "Have students scan problems to determine which will require computation and which ones require comparison."

Eight pages later:

> "For each problem, ask students to explain their choice of a computation method."

Multiplication. Twenty-one pages later, on the first lesson on which multiplication problems appear:

> "Have student read all problems before actually solving any of them. Ask which problems require finding an estimated product."

Division. Twenty-seven pages later, on the first lesson on which division problems appear:

> "For each problem, ask students to explain their choice of a computational method."

Connecting Math Concepts devotes a great deal of time to *teaching* students explicit strategies for solving word problems. Because of the complexity of teaching word problems, the strategies cannot be described adequately in this article. However, descriptions are available in the next wo articles. The effectiveness of approaches such as these has been found in research with multiplication and division word problems (Darch, Carnine, & Gersten, 1984; Gleason, Carnine, & Boriero, in press), and with ratio word problems (Moore & Carnine, 1989).

Guided and Independent Practice

As noted earlier, traditional basals offer rather vague explanations for introducing new concepts. After these initial explanations and activities, students are expected to work several problems on their own, without explicit guidance from the teacher.

Many students need a transition between the explanation given in the introduction and the problems to be worked independently. Good, Grouws, and Ebmeier (1983) found that guided practice is an effective way for teachers and students to interact. In guided practice, which occurs after a concept is introduced, the teacher asks questions that prompt appropriate student application of the new concept. In *Connecting Math Concepts*, guided practice might include these questions to guide students in completing their practice problems on borrowing. "Are you going to start in the ones column or tens column?...Read the problem in the ones column. ...Is the bigger number on top?...So do you need to borrow?...How do you do that?..." These questions are repeated for three or four problems, which reminds the students of where to start working, whether to borrow, and how to borrow.

Guided practice is the primary means by which the teacher insures that the students can apply the concepts they learn. During guided practice, teachers prompt the students, but as the students approach mastery, teachers should decrease the level of prompting until the students are functioning independently (Paine, Carnine, White, & Walters, 1982).

Practice should also continue after a concept is introduced so that the students will remember how to apply it when it is integrated in a more complex concept. For example, in the basal program that best teaches fractions, the skill of finding the least common multiple was introduced in one lesson, neglected for the next seven lessons, reviewed in one lesson, neglected again for six lessons, and then reappeared in the context of adding and subtracting fractions with unlike denominators. This teaching sequence amounted to two exposures over 15 lessons, which is not sufficient teaching or review for even average-ability students.

In *Connecting Math Concepts*, the important and complex skill of least common multiple is practiced on every lesson, before it is subsumed in adding and subtracting fractions with unlike denominators.

Independent practice needs to encompass a sufficient number of examples and span enough lessons so that a full spectrum of students will have ample opportunity to become proficient. The inadequacy of practice in the early grades is clearly illustrated with instruction on basic facts. Figure 6 indicates that students are expected to learn all their addition facts with 10 lessons of practice, covering 411 practice examples, which is an average of about four tries on each of the 100 facts. The situation is similar for subtraction—9 lessons with 259 practice examples, less than three tries on each of the 100 facts. Moreover, there are long sequences of lessons with little or no review. Half the chapters have 20 or fewer addition practice problems. In more than half the chapters, there are no subtraction practice problems.

In *Connecting Math Concepts,* all addition and subtraction facts are not introduced in second grade. (Carrying and borrowing problems are made up of only familiar, previously introduced facts.) Facts are practiced every lesson throughout the program. There are no gaps in practice as found in the basal described in Figure 6.

Insufficient review can have disastrous consequences. A principal of an elementary school wanted to ensure that all students learned the multiplication facts. He made learning multiplication facts the focus of a school-wide effort in the fall—charts were placed around the school; inter-room competitions were conducted and so on. Within two months, almost every student was proficient. But the multiplication facts were not reviewed in the winter and spring. The following fall, the principal found very little retention of the multiplication facts. He was surprised and disappointed. *With a few minutes of review* each day during the winter and spring, the principal's program would have been a success.

Figure 6. Addition and Subtraction Practice Problems in Typical Second Grade Math Textbook

	Addition only		Subtraction only		Mixed		
Chapters	Number of Lessons	Number of Problems	Number of Lessons	Number of Problems	Number of Lessons	Number of Problems	Activity
1. Addition and Subtraction to 12	3	113	4 0	114	1 1	50 29	Fact Families Mixed Practice
2. Place Value to 99	0	0	0	0	0	0	
3. Addition Facts to 18	7	298	0	0	0	0	
4. Subtraction Facts to 18	1 ((missing addends))	8	5	145	1 1	58 5	Fact Families Story Problems
5. Time, Money, Msm't	2	11	3	20	1	6	Choosing Operation
6. Additon of 2-digit Numbers	8	178	0	0	1	4	Story Problems
7. Subtraction of 2-digit Numbers	1	2	4	142	2 1	17 34	Problem Solving 15-Sub., 2-Add. Checking the Answers
8. Geometry, Fractions	1	5	0	0	0	0	
9. Time, Money, Msm't	0	0	0	0	0	0	
10. Place Value to 999	1	2	0	0	0	0	
11. Adding & Subtracting 3-digit Numbers	5	130	3	119	1 1	8 6	Estimation, More/Less Problem Solving
12. Multiplication & Division	1	10	0	0	1	10	Real Integrated Practice

Appropriate Examples

The amount of practice and review is not the only important aspect of examples. The quality of the examples is also crucial. Hamann and Ashcraft (1986) reviewed the presentation of basic math facts in widely used mathematics textbooks from kindergarten through third grade and found "... the frequency of occurrence distributions for the basic facts were markedly skewed" (p. 173). In particular, they found that "there were many fewer presentations of large than small problems in the texts, and problems involving the addition of zero were relatively infrequent at all grades" (p. 173).

Hamann and Ashcraft further noted a clear relationship between the observed distribution of problems and the ease with which problems were solved: Facts with larger problems and with zero that appeared least in texts were also the most difficult facts, as measured both by response time and error rates for students from first grade through college. Students need extra practice on more difficult content, not easier content.

The quality of examples involves more than deciding which ones to emphasize. Inappropriate examples must be avoided to prevent confusion in the students (Carnine, 1980b). For example, one basal program suggested that students multiply to find the perimeter of a square. While multiplication to determine the perimeter of a square is valid, students will later learn to multiply to find the *area* of rectangles; some students will become confused and will multiply to find the perimeter of rectangles that are not squares. Using addition for perimeter and multiplication for area is much safer for third graders.

Another example is the basal introduction of fractions as parts of one whole—1/3, 2/3, 3/3, 1/4, 2/4, etc. The next year, students encounter mixed numbers with only "proper" fractions in the basal; that is, the fractions are still less than one part of a pie. As a result, students have had at least two years to become convinced that a fraction always represents a portion of a pie; all fractions are the same in that they represent part of a whole. In the third year, students typically encounter improper fractions. This is especially bewildering to low-performing students, who predictably apply what they learned previously (that a fraction is part of one pie). As a result, the students will likely draw this picture to represent 4/3:

As demonstrated earlier, fractions *can* be carefully introduced so that students understand that fractions can represent more than one whole.

Another case of inappropriate examples is suggested by findings from the National Assessment of Educational Progress (Carpenter, Coburn, Reyes, & Wilson, 1976). Many students apply an unintended rule about denominators when adding fractions (i.e., do what the sign says). For example, students might give 2/5 as the answer for 1/3 + 1/2. The fallacious rule comes from students' experiences with whole numbers and with multiplying fractions. With whole numbers, students always act on the numbers; e.g., 3 + 2 = 5. Similarly, when students multiply 1/3 x 1/2, the numerators and the denominators are multiplied. Students then apply the unintended rule—"operate on the denominators"—to addition problems (1/3 + 1/2), and mistakenly add both the numerators and the denominators to get 2/5.

Most basal programs unintentionally promote this misrule. They teach adding and subtracting fractions in one chapter and multiplying and dividing fractions in a different chapter. The programs *never give integrated practice.* Because of the lack of integration of addition and multiplication of fractions, students do not receive any explicit instruction or guided practice in distinguishing fraction addition from fraction multiplication. This misrule is an example of how students' understanding of whole number concepts and operations interferes with their understanding of fractions (Behr, Wachsmuth, Post, & Lesh, 1984).

Conclusion

Good, Grouws, and Ebmeier (1983) noted that "insufficient attention has been given to the quality of development in our work and in educational research generally" (p. 199). They view development as a "collection of acts controlled by the teacher" (p. 207) that consists of five components: (a) attending to prerequisites, (b) attending to relationships, (c) attending to representation, (d) attending to perceptions, and (e) attending to the generality of concepts. In their research, Good and Grouws pointed out that development "appears to be the only variable that teachers, as a group, had consistent trouble in implementing" (1979, p. 358).

A close look at traditional basals suggests that publishers are not meeting their responsibilities to assist teachers in providing suitable development for students. The content of mathematics is extensive, often difficult and interrelated in complex ways. If basal texts do not deal with these aspects of the mathematics curriculum, how can teachers, who have little "free time," be expected to systematically address Good's et al. (1983) five components? Improving mathematics performance will not be possible without reforming the math textbooks that define the curriculum. The data from the Direct Instruction Model suggest that the following reforms will help prevent many students from failing in math:

1. Organize lessons around strands, not a single topic.
2. Design lessons to maximize instructional time so that all students have an opportunity to learn and apply important concepts.
3. Introduce concepts at a reasonable rate.
4. Create explanations and activities that clearly communicate new concepts, leading to both understanding and proficiency.
5. Provide guided and independent practice.
6. Select appropriate examples.

Math curricular material designed with these guidelines in mind will relieve much of the unfair burden placed on teachers for development. Curricular material should provide field-tested suggestions for developing understanding and proficiency. Such material will benefit both teachers and students. ◆

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Making Connections in Third Grade Mathematics: *Connecting Math Concepts*

by Douglas Carnine
Siegfried Engelmann
University of Oregon

The greatest challenge in teaching mathematics in the intermediate grades is not developing computational proficiency, a necessary goal, but instilling an integrated schema for mathematics. When students can see the key relationships within mathematics, they will be able to make important connections. This paper illustrates this process with the third grade level of *Connecting Math Concepts* (Engelmann and Carnine, 1991).

Although *Connecting Math Concepts* is just being printed, some evaluations have been carried out, one with two very good third-grade teachers with predominately low-income, minority students. The mean achievement was 5.6 grade level, far above the expected score of these students at the end of third grade. (In a third grade rural school in another state, the mean percentile in methematics for the 26 third graders in *Connecting Math Concepts* was 79.

Another study compared four high-performing third graders in *Connecting Math Concepts* with four high performers in a conventional basal. One aspect of the investigation had to do with solving types of problems the children would not have encountered in their textbook. For example:

> 104 fifth graders are taking two buses on a field trip. Fourth graders can go in the extra seats. The bus leaving from the north end of town holds 72. The bus leaving from the south end of town has 14 fewer seats. Fifity-five fifth graders will get on the bus at the north end of town. How many fourth graders can take the bus at the north end of town? How many fourth graders can take the bus at the south end of town?

All third graders in *Connecting Math Concepts* solved the problem; none of the students in the other program did. Another aspect of that evaluation looked at the degree to which the students were able to make connections between various math concepts. (Some of these connections are illustrated later in this article.) The third graders in *Connecting Math Concepts* saw 50% more relationships among math concepts than the comparison students.

A final evaluation looked at how well the problem solving strategies taught in Connecting Math Concepts transfered to "real life" problems presented on video. The *Connecting Math Concepts* students solved about 80% of the real life problems, while the comparison group solved about 50%.

Addition and Subtraction Facts

Basic Facts as Families

Addition and subtraction facts are usually treated as 200 discrete sets of three numbers to be memorized. In *Connecting Math Concepts*, facts are treated as interrelated concepts—members of number families. This structure prompts important relationships between addition and subtraction, as well as reduces the number of sets to be memorized from 200 to 55. From these 55 number families, all 200 addition and subtraction facts can be quickly derived. The 55 number families appear in Figure 1 (see page 18).

Number families are written on an arrow so that they can be transformed into both addition and subtraction statements. For example, in the number family $\xrightarrow{6 \quad 9} 15$, 6 and 9 are treated as "small numbers" and 15 is is treated as a "big number." Four facts can be derived from this family. Each of two addition facts starts with one of the small numbers and adds the other small number to produce the "big" number: $6 + 9 = 15$ and $9 + 6 = 15$. The subtraction facts begin with the big number and subtract one of the small numbers, yielding the other small number: $15 - 6 = 9$ and $15 - 9 = 6$. Because 45 number families lead to two addition facts and two subtraction facts, students who memorize 55 number families know how to deduce all 200 facts: 45 families times 4 facts (2 addition and 2 subtraction) equals 180 facts. The ten families on the diagonal in Figure 1 (e.g., $\xrightarrow{1 \quad 1} 2$, $\xrightarrow{2 \quad 2} 4$, $\xrightarrow{3 \quad 3} 6$) yield 20 facts (e.g., $1 + 1 = 2$; $2 - 1 = 1$; $2 + 2 = 4$; $4 - 2 = 2$; etc.). Memorizing 55 families is easier and less time consuming than memorizing 200 facts. Teaching the fact number families also promotes the integration of the concepts of addition and subtraction.

The Counting Relationships Among Facts

More can be done to help students learn facts than just reducing the memory load from 200 sets of numbers to 55. Additional teaching of counting relationships between facts eases the learning of these 55 sets. For example, number families with a 1 are fairly easy to learn, because the *big* number is the next number, when counting in order. For example, 8 is the big number after 7 when counting by 1: $1 + 7 = 8$. Similarly, 9 is the big number after 8 when counting by 1: $1 + 8 = 9$. The big numbers in the top row of

Figure 1. Addition and Subtraction Number Family Table

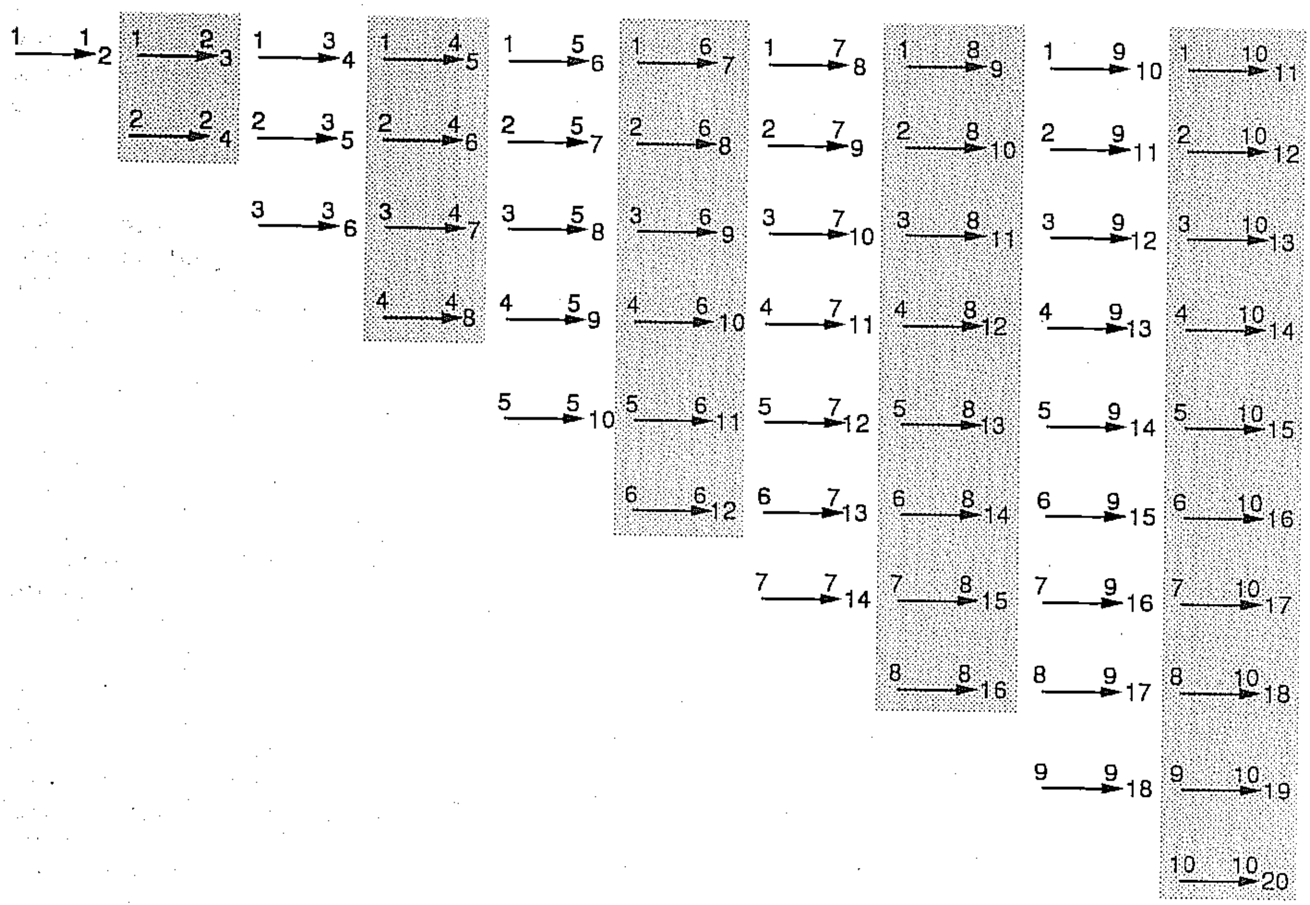


Figure 1 are simply the counting numbers 2, 3, 4, 5, 6, 7, 8, 9, 10.

Facts with 2 as a small number are closely related to facts with the number 1 as a small number. This pattern is reflected in these corresponding addition facts:

$$+1\left(\begin{array}{c}1+6=7\\ \\ 2+6=8\end{array}\right)+1$$

This relationship is repeated for every pair of families. The increment from 1 to 2 equals the increment from 7 to 8 (and from 8 to 9, from 9 to 10, and so forth):

$$+1\left(\begin{array}{c}1+7=8\\ \\ 2+7=9\end{array}\right)+1$$

This relationship helps students learn to go from the easier facts with a 1 to facts with a 2; if *1* + 6 = *7*, then 2 + 6 = *8*.

Another example of counting relationships among facts can be seen in the relationship between facts with 10 as a small number and the difficult group of facts that have 9 as a small number. Addition facts that contain the number 10 as a small number are easy to remember, because the digits in the answer come from the digits in the added numbers. For example, in the problem 3 + 10 = ☐, the digits for the answer appear in the added numbers 3 + 10 = 13. The answer 13 is composed of one ten and three ones.

The simplicity of facts with a 10 can help students with the more difficult group of facts, those with a 9. For example, the family of 3, 10, 13 in the last column of Figure 1 has the corresponding family of 3, 9, 12 in the preceding column. The two corresponding addition facts for these families are:

9 is one less than 10

3 + 9 = 12   3 + 10 = 13

so 12 is one less than 13

This relationship applies to every pair of problems:

9 is one less than 10

$4 + 9 = \square \qquad 4 + 10 = 14$

so $\square$ is one less than 14

When students see any addition problems with a 9, such as $7 + 9$, they can think of the easy corresponding fact ($7 + 10 = 17$) and come up with the answer to $7 + 9 = \square$, which is one less than 17.

$$7 + 10 = 17 \quad \text{so} \quad 7 + 9 = \boxed{16}$$

For research on the relationship between counting strategies and fact acquisition, see Carnine and Stein (1981), Carpenter and Moser (1984), and Thornton (1978).

Problem Solving

The most prevalent and frustrating math application in primary- grade math programs is word problems (Kameenui & Griffin, 1989). The frustration stems largely from an inability of mathematics educators to devise explicit strategies that a full spectrum of students can learn and successfully apply. A consequence of this frustration is the avoidance of any but the most rudimentary type of addition and subtraction word problems in textbooks (Peterson, Fennema, & Carpenter, 1988). Peterson, et al., recommend that students be prepared to handle the variations of the four basic types of word problems: *join, separate, compare,* and *part/part-whole* (part/part-whole refers to classification—*cats* and *dogs* would be the parts and *pets* would be the whole). The following analysis illustrates explicit strategies that students can successfully apply to this full range of problems.

An explicit strategy should prepare students to see the total structure of a problem and *not* just rely on specific key words. For example, the word "more" appears in many joining problems calling for addition, e.g., "Juan had 7 marbles. He won 5 *more.* How many marbles does he have now?" But the word also appears in a significant number of comparison problems that call for subtraction: Jill had 614 dollars. Tom had 829 dollars. How much *more* money did Tom have? Students who think the word *more* always represents joining and calls for addition have a superficial understanding of problem solving, at best.

The strategy illustrated next is aimed at teaching students to see the relationship between the situation described in a story problem and the concept of a number family composed of two small numbers (e.g., 5 + 4) and a larger number (e.g., 9). Number families are useful because they provide a map that can be used to diagram the various types of word problems. The number family map in turn leads to setting up the addition or subtraction calculation. The strategy teaches students not to make quick judgments because of the presence of a particular word such as *more.*

The strategy has students work in two stages. The students first graphically represent the situation described in the word problem; second they determine how to write the number problem. The strategy will be illustrated with *joining* and *separating* problems, then *comparison* problems, and finally *part/part-whole* problems.

Joining and Separating Problems

In the word problem below about Marco, special-needs students are likely to add 57 and 112, because the problem says that Marco saved dollars. The students assume that because of the word *saved*, they should carry out the operation for joining by adding the numbers that are given. However, adding 57 and 112 does not lead to the correct answer.

> Marco's mother will give him some money for a school trip. He already has saved 57 dollars. He needs 112 dollars. How much money will his mother give him?

To prevent this confusion, teachers explain how to represent the joining situation described in the word problem. This representation takes the form of a diagram based on the number family analysis. In joining problems, such as the one about Marco saving money, the numbers that are joined, or added, are the first numbers. The total, in this case the number of dollars Marco needs, is the big number. As students have learned from working with number families, the two small numbers go on top of the arrow and the big number, which is the sum in this example, goes at the end of the arrow :

$$\xrightarrow{\square \quad 57} 112.$$

After students represent the situation with a number family, they are ready to apply what they have learned about the relationships between addition and subtraction to compute the answer. For example when the unknown in a number family is a small number, such as :

$$\xrightarrow{\square \quad 57} \quad 112$$

the number family can be translated into a subtraction problem. The big number, 112, is the first number in the subtraction problem. The small number that is given, 57, is then subtracted from the big number ($112 - 57 = \square$) to produce the other small number: $112 - 57 = 55$.

In short, students learn first to represent the joining situation described by the word problem and, second, to decide how to compute the answer based on the relationship between addition and subtraction. For example, if both small numbers are given in a problem, they are written above the arrow.

$\xrightarrow{32 \quad 89} \square$

The students then add the small numbers to figure out the big number:

$$\begin{array}{r} 32 \\ \underline{+\,89} \end{array}$$

Comparison Problems

In comparison problems, the comparison can be information given in a problem (e.g., Marco sold 57 fewer subscriptions than Lui) or the unknown asked about in a problem (e.g., How much heavier was Mary?). Because of the words "sold fewer" in the following problem, many special-needs students will subtract.

> Marco sold 57 fewer magazine subscriptions than Lui. Marco sold 112 subscriptions. How many subscriptions did Lui sell?

Again, students can use number families to keep from getting confused. The first step is to represent the problem using a number family; students must identify the two numbers stated in the problem as either both small numbers or a small number and the big number. The students are shown a simple way to do this: They find the sentence that tells about the comparison and read it without the number that tells how many more or how many less. For example, students are taught to read the first sentence without the 57: "Marco sold fewer subscriptions than Lui." Because Marco sold fewer subscriptions, Marco is represented by a small number. By default, Lui is the big number. The students write M for Marco and L for Lui:

$\xrightarrow{\quad M} L$

The word problem also has a number that gives the difference between Marco and Lui. That number is always a small number. Marco sold 57 fewer, so 57 is the other small number:

$\xrightarrow{57 \quad M} L$

The students next read the rest of the problem. The problem asks about Lui and gives a number for Marco, so the students draw a box around L and cross out Marco and write 112:

$\xrightarrow{57 \quad \overset{112}{\cancel{M}}} \boxed{L}$

Because the problem states both small numbers, the students write an addition problem:

$$\begin{array}{r} 57 \\ \underline{+\,112} \end{array}$$

The answer tells how many magazine subscriptions Lui sold.

Part/part-Whole Problems

Part/part-whole problems, can be thought of as classification problems. To work part/part-whole problems, students need to understand the relationships among the classes named in a problem. For example, cats and dogs are members of the larger class, pets; or as in the following problem, magazine subscriptions and newspaper subscriptions are members of the inclusive ("larger") class, subscriptions. As in the earlier examples, special-needs students are likely to miss the problem by adding the numbers, because of the verb "get."

> Maria has to get 112 magazine and newspaper subscriptions. She is sure she can get 57 magazine subscriptions. How many newspaper subscriptions does she have to get?

To represent this situation with a number family, students treat subscriptions, the *big* class, as the *big* number. The names for the subordinate classes go in the places for the "small" numbers, on top of the number family arrow:

$\xrightarrow{\text{magazine} \quad \text{newspaper}} \text{subscriptions}$

The students cross out the words that have number values and draw a box around the word the problem asks about:

$\xrightarrow{\overset{57}{\cancel{\text{magazine}}} \quad \boxed{\text{newspaper}}} \overset{112}{\cancel{\text{subscriptions}}}$

The students know the big number, 112; so they subtract: $112 - 57 = \square$. The answer gives the number for newspaper subscriptions.

Students work many types of word problems besides those requiring addition and subtraction of whole numbers. For example:

Multiplication and Division

a. There were 8 rooms. Each room had 9 tables. How many tables were there in all?
b. Each dog had 5 bones. There were 45 bones in all. How many dogs were there?

Estimation, Geometry, and Multiplication

For these problems, students use estimatation to draw a proportional representation of the figure described in the problem. Then they write the length for each side. Then they write the multiplication problem and the answer:

a. A floor is the shape of a rectangle. It is 10 feet long and 8 feet wide. What is the area of the floor?
b. A large field is the shape of a rectangle. It is 3 miles long and 5 miles wide. What is the area of the field?

Coins and Cents

a. Tom had nickels and dimes. Tom had 40 cents in nickels and 7 dimes. How many cents did Tom have in all?
b. Alice had pennies and quarters. She had 20 cents in pennies and 70 cents in all. How many cents did she have in quarters?

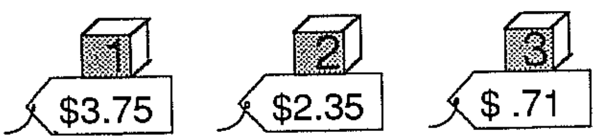
Fractions with Like Denominators

a. A kitten weighed $\frac{8}{3}$ pounds. Then the kitten gained $\frac{2}{3}$ pounds. How many pounds was the kitten?
b. A bag of nails weighed $\frac{9}{4}$ pounds. Somebody took $\frac{5}{4}$ pounds of nails from the bag. How many pounds were left in the bag?

For research on teaching students explicit strategies to solve word problems, see Darch, Carnine, and Gersten (1984), Gleason, Carnine, and Boriero (in press), and Moore and Carnine (1989).
Problem solving is not restricted to word problems. Some other activites are shown below.

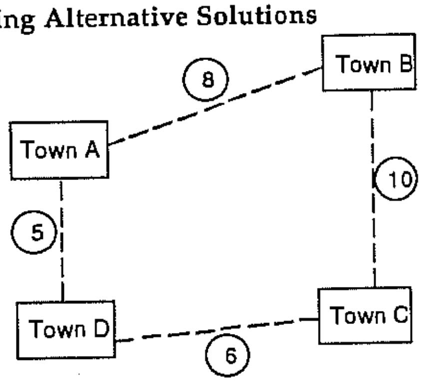
Identifying Relevant Information

Read each problem to see what the person buys.
Add up only those amounts.



a. A person buys items 1 and 3.
How much does the person spend?

Evaluating Alternative Solutions



Don and Dan went from town A to town D.
Don said, "The trip is less than 10 miles."
Dan said, "The trip is more than 10 miles."

a. Which person went through town B? ______
b. How many miles did that person travel? ______
c. Which person did not go through town B? ______
d. How many miles did that person travel? ______

Determining the Correct Operation

a. $5 + 7 = 4\square 3$ b. $3 \times 5 = 18\square 3$

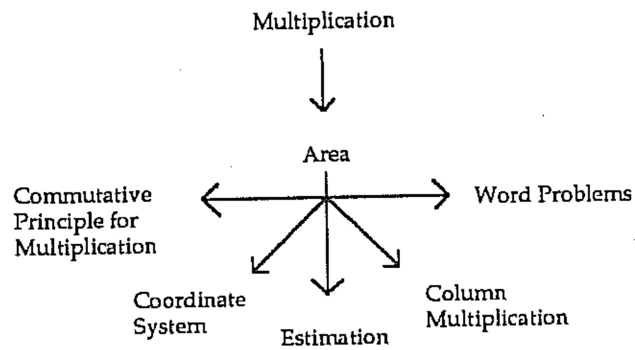
Determining the Missing Value

a. $2 + 8 + 4 = \square + 4$ b. $6 + 8 + 2 = 17 - \square$

Other Linkages

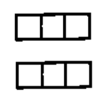
Teaching connections in mathematics is important for not only problem solving; important linkages should be made among all major concepts. Multiplication serves as an example of how these linkages are made in *Connecting Math Concepts* (Engelmann and Carnine, 1991). The way in which the concept of multiplication is introduced and related to other concepts is summarized in Figure 2.

Figure 2. Concepts Developed by Building from the Familiar to the Unfamiliar



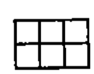
Multiplication

At the top of Figure 2 is multiplication, which is based on counting. For example, to introduce 3 x 2, students create or are shown two rows of blocks, each with three blocks.



The teacher says, "I can figure out how many blocks by counting a fast way. There are 3 blocks in each row, so I count by 3 for each row: 3, 6. Let's see if the fast way works. You count the blocks one at a time: 1, 2, 3, 4, 5, 6." Students then write multiplication equations, such as 3 x 2 = 6. Later they work from pictures of columns or blocks and eventually work symbolic problems without pictorial representations.

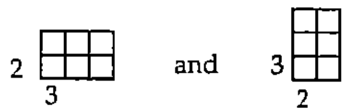
Area. Next, the rows of blocks are joined. Rather than two separate rows of blocks, students see this figure:



Students are told that they can use multiplication to figure out how many squares are in the figure.

Three squares in each row and two rows, or 3 x 2. The area of the figure is six square units.

Commutative Property for Multiplication. The bridge to the commutative property for multiplication, which is important in teaching multiplication facts, occurs in this way. Students are shown



and are told that the figures have the same area; the second figure is just turned up on its end so the 2 is on the bottom. The students write a statement for each figure; 3 x 2 = 6 and 2 x 3 = 6. Both figures have the same number of squares and therefore the same area. The figures illustrate that the answers for 3 x 2 and 2 x 3 are the same. Subsequently, when students learn the answer to 6 x 8, for example, they realize they also know the answer for 8 x 6. These exercises lead naturally to multiplication and division number families.

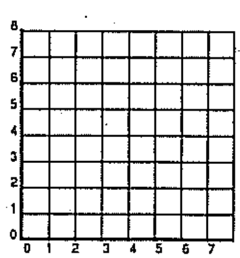
Students learn that there are number families for multiplication and division analogous to those for addition and subtraction. This new number family resembles a division problem. Just as is the case for an addition and subtraction number family, a multiplication and division number family yields four facts—two multiplication and two division. For the family:

$$\begin{array}{r} 8 \\ 2\overline{)16} \end{array}$$

the facts are:

$$\begin{aligned} 2 \times 8 &= 16 \\ 8 \times 2 &= 16 \\ 16 \div 2 &= 8 \\ 16 \div 8 &= 2 \end{aligned}$$

Coordinate System. The coordinate system provides reference numbers for any point on a 2-dimensional grid. When one corner of a rectangle is placed at the origin of a 2-dimensional grid (0,0), the opposite corner of the rectangle is represented by the coordinates for that point. For example, the corner of a 4 x 6 rectangle has coordinates of x = 4 and y = 6.



Students are simply told that the coordinate system gives a code for drawing rectangles. The students start at a point called the origin. The code for how wide to make a rectangle is the number given for the letter x. The code for how high to make a rectangle is the number given for the letter **y**. The students work from values for x and **y** to identify the far corner of the rectangle. From that point, they draw the sides of the rectangle, and calculate its area. Later, students are shown a point, and they write the x and **y** values.

Estimation. Estimation is often difficult for students because they don't have the frame of reference for "guessing intelligently," a prerequisite for checking the reasonableness for their answers. Introducing estimation in the context of area provides a good demonstration of how to guess intelligently. In the introductory estimation exercises, students use a ruler to draw a side of a rectangle (e.g., 4 inches wide). The students draw the next side (e.g., 5 inches) without the ruler. They use the 4-inch line as a basis for estimating the length when drawing the 5-inch side. The frame of reference for estimation comes into play because the 5-inch side they make without the ruler is slightly longer than the 4-inch side. The students can visually check the reasonableness of their answer. They can see if the rectangle they drew is a little taller than it is wide. They then multiply to calculate the area.

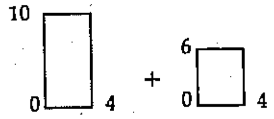
Another estimation activity provides both dimensions for one rectangle, but just the dimensions for a second rectangle:

4 x 6 3 x 7

The students use estimation to draw a second rectangle that is supposed to be 3 inches wide and 7 inches high. In this exercise, the basis for estimation is the original rectangle. The new rectangle should be a little narrower, but a little higher, than the original rectangle. Again students can visually check the reasonableness of their answer. In both estimation exercises, students are learning a new skill, estimation, in the content of a familiar skill, area.

The estimation introduced with these rectangle activities is extended in several ways. As is typically done, children estimate the answers to computation problems. However, they also estimate amounts from menus and catalogs and then compare their estimates with exact calculations.

Column Multiplication. Area can also be used in teaching column multiplication. In introductory exercises, students are shown how to calculate the total area for two figures that both have the same width (e.g., 4' x 10' and 4' x 6').



Students figure out the area for each rectangle by constructing simple multiplication facts. They then add to find the total area, 40 + 24 or 64.

$$\begin{array}{r} 10 \\ \underline{\times 4} \\ 40 \end{array} \qquad \begin{array}{r} 6 \\ \underline{\times 4} \\ 24 \end{array}$$

Next column multiplication is introduced as a short cut for figuring the area of any two rectangles with a side the same length. The students write the width, e.g., 4, just one time. Initially the heights for the two rectangles—10 for the first and 6 for the second—are written as an added number (10 + 6):

$$\begin{array}{r} (10 + 6) \\ \underline{\times \quad 4} \\ 24 \\ \underline{+\ 40} \end{array}$$

The students first multiply 4 x 6, then 4 x 10. Next, the students are shown that the 10 + 6 can be written as 16; the problem, though, is still worked by multiplying 4 x 6 and 4 x 10, then adding the products:

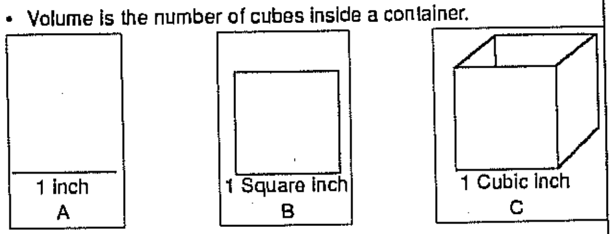
$$\begin{array}{r} 16 \\ \underline{\times \quad 4} \\ 24 \\ \underline{+\ 40} \\ 64 \end{array}$$

At this point students are working column multiplication problems.

Volume. An obvious extension of area is to volume. Multiplication takes into account a third dimension. (Figure 3.)

Figure 3.

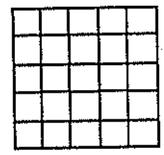
- Volume is the number of cubes inside a container.



- Every box has height, width, and depth.
- The depth is how far back it goes.
- To find the volume of a box, you multiply height times width times depth.
- The units in the answer are cubic units not square units.

Word problems. A final skill that can be integrated into students' prior knowledge of area is the introduction of multiplication and division word problems. The slightly more advanced problem types that would appear after this introduction were illustrated earlier in the section on problem solving.

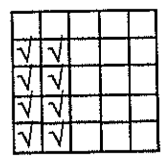
Multiplication and division word problems can be introduced in exercises in which students work with blocks or a coordinate system grid:



An early word problem might tell about squares on a grid (or blocks). A typical problem follows:

> A rectangle has two squares in each row. There are eight squares in all. How many rows of squares does the rectangle have?

The students draw a line under two squares on the bottom of the grid to show how wide the rectangle is. Next, they count the squares two at a time (2, 4, 6, 8), marking a completed row each time they count, until they reach 8. They then can see the number of rows, 4.



Students learn to solve word problems without a grid or blocks next and then problems involving a variety of objects and events, such as the problems illustrated earlier with dogs and their bones and tables in rooms.

Collecting and Analyzing Data

A major objective of the new National Council of Teachers of Mathemtics standards is to teach students to reason mathematically. One important way to reason mathematically is to collect and analyze data. An early step involves interpreting data in a table, as exemplified in the questions for the table below. (Figure 4.) Next, students begin operating on the data in a table by adding to determine the totals. (Figure 5.)

At the next stage, students learn to fill in blank cells in a table, using the number family analysis. (Figure 6.) For example, in the top row, a small number and a big number are given, so the students start with the big number and subtract: 11 – 7 = □. The number that goes in the blank cell in the top row is 4. Students later solve for the numbers that go in the other blank cells and answer questions based on the complete data in the table. (Figure 7.)

Figure 4.

Rainfall During 3 Months

	May	June	July	Totals for both months
River City	6	9	1	16
Hill Town	3	1	8	12
Oak Grove	0	7	9	16
Total for all cities	9	17	18	

a. Which month had the most rainfall?

____

b. Which month had the least rainfall?

____

c. Which city had the least amount of rain? __________

d. How much rain fell in all the cities during July? __________

Figure 5. TheNumber of Cars That Went Down Different Streets.

	Elm Street	Oak Street	Maple Street	Totals
Red cars	4	5	9	
Yellow cars	2	2	8	
Blue cars	4	4	1	
Totals				

Figure 6.

Write the problems for each row and the answer. Then write the missing numbers in the table. Add the totals for the rows to figure out the total total and check it by adding the totals for the columns.

			totals	
a.		7	11	__________
	8	3		__________
		2	7	__________
totals				

Figure 7.

This table shows the number of big cars and small cars that parked in lot A and lot B.

	Lot A	Lot B	Total
Big cars	7	5	
Small cars		6	
Total in lots	11		22

a. How many small cars parked in lot A? __________

b. How many big cars parked in the lots? __________

c. How many big cars and small cas parked in the lots? __________

d. How many big cars parked in lot B? __________

In later lessons, students are given data that they insert in a table. Then they solve for the blank cells and answer questions based on the table. (Figure 8.) Later, the students collect their own data and analyze it to answer questions they generate.

The data analysis is extended to tables involving time, in which students work with departure times, duration and arrival times. (Figure 9.) Again, students insert data into the table, solve for blank cells and answer questions based on the table.

Figure 8.

	Mountain Park	Yellow Park	Totals
Small rocks			
Large rocks			
All rocks			

Fact A: In Mountain Park, there are 18 rocks in all.
Fact B: In Mountain Park, there are 10 small rocks.
Fact C: The total number of rocks in the both parks are 35.
Fact D: In Valley Park, there are 5 large rocks.

a. Were there more large rocks in Mountain Park or Valley Park?

____

b. How many large rocks were there in both parks? ______

c. Were there more large rocks or small rocks? __________

d. Were there more rocks in Mountain Park or Valley Park?

A new analysis is introduced for data with multiplicative relationships. (Figure 10.) Given the cost of five ounces, students determine the cost of one ounce, which involves the concept of unit pricing.

Figure 9.

Fill in the missing time and answer the questions.

	Time left	Minutes of Trip	Time arrived
a. Fran	5:09		5:46
b. Ana		:41	7:56
c. Dan	5:19	:12	
d. Diane			
e. Roxanne			

d. Diane left for the party at 5:15. The trip took 38 minutes. When did she arrive at the party?

______

e. Roxanne left for the party at 5:12. She arrived at 5:31. How long did the trip take? ______

Figure 10.

Fill in the table. Then decide which brand of corn cost the least for each ounce.

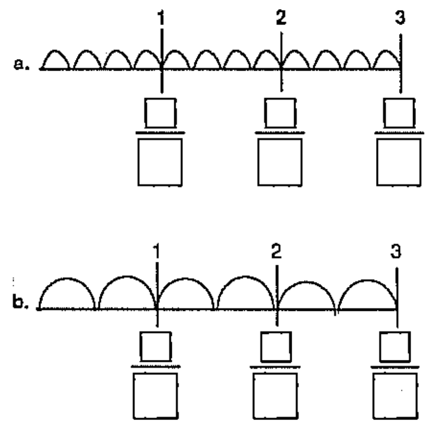
	Cents for 1 ounce	Cents for 5 ounces
Brand A		20
Brand B		35
Brand C		30

Fractions

The most important linkage for fractions is with whole numbers. Students must understand how fractions and whole numbers form a single number system. Fractions is not some isolated type of number skill. This linkage is done by developing fraction concepts on portions of a ruler, which is a number line that accommodates both fraction numbers and whole numbers. Students first write the equivalent fraction for each whole number on a picture of a ruler. The students count the number of parts in each inch, and write that number as the denominator in each fraction. They then count all the parts to the first inch marker and write that number as the first numerator. Next, they count all the parts to the second inch marker and write that number. (Figure 11.) The ruler representation for fractions intentionally introduces the concept that fractions can be more than one. Typically this aspect of fractions is withheld from students, which causes confusion when it is eventually introduced.

Figure 11.

Write the fraction for each inch.



In later lessons, students are given a picture of a fraction and write the numerical fraction. This translation involves representations that are number lines (not necessarily in inches) and that are geometric figures. (Figures 12 and 13.)

Figure 12.

Write the fractions.

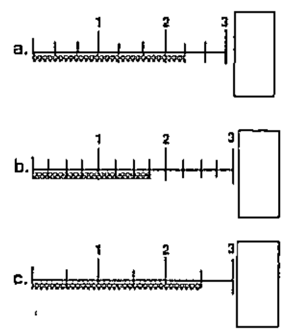
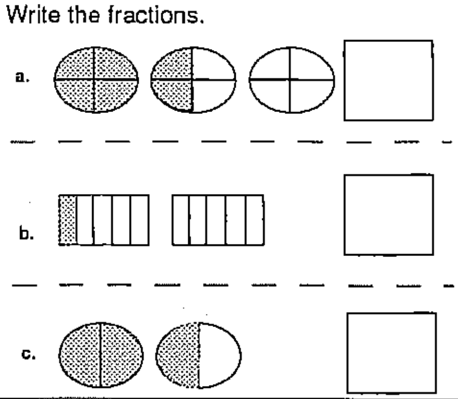


Figure 13.

Write the fractions.



Once students understand the basic concept of a fraction, they evaluate fractions as being more than one, equal to one, or less than one. This evaluation is confirmed by drawing a picture of the fraction. (Figure 14.)

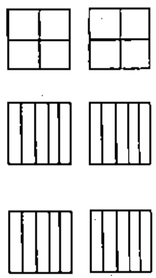
Figure 14.

Circle **more than 1, less than 1** or **equals 1.** Then shade the parts of the picture.

a. $\frac{4}{4}$ More than 1 / Less than 1 / Equals 1

b. $\frac{4}{5}$ More than 1 / Less than 1 / Equals 1

c. $\frac{5}{5}$ More than 1 / Less than 1 / Equals 1



Next, students add and subtract fractions. They must first recognize that these operations can only take place if the fractions have the same number of parts. It's analogous to trying to add apples and oranges. Students have to think of them differently, as pieces of fruit. Then addition is possible. Students eventually will learn to think of thirds and fourths differently, as 12ths. At this point, however, students are expected only to recognize that addition and subtraction of fractions require the same denominators (Figure 15).

Figure 15.

Draw a line through the problems you can't work. Then work the rest of the problems.

a. $\frac{3}{4} + \frac{2}{3} = \_\_\_\_ = \_\_$

b. $\frac{3}{10} + \frac{9}{10} = \_\_\_\_ = \_\_$

c. $\frac{17}{5} - \frac{8}{9} = \_\_\_\_ = \_\_$

d. $\frac{18}{3} - \frac{9}{3} = \_\_\_\_ = \_\_$

As mentioned earlier, students learn to relate fractions to whole numbers by writing a fraction for each whole number. They determine the fraction to write by counting the parts on the number line. The relationship between fractions and the corresponding whole numbers is an important concept in more advanced work with fractions—simplifying fractions. When students are expected to write 12/3 as 4, the students are expressing the equivalency between those two values on the number line. At this more advanced stage of work with fractions, always counting on a number line is too slow and cumbersome. Students need to learn the relationship between fractions and multiplication. This instruction entails rotation of the multiplication number families to express the relationship between fractions and whole numbers. Here are several examples where the number family is rotated 90% so that the arrow points up.

$2 \xrightarrow{9} 18$ becomes $\frac{9 \mid 18}{2}$ which is the same as $\frac{18}{2} = 9$

$5 \xrightarrow{7} 35$ becomes $\frac{7 \mid 35}{5}$ which is the same as $\frac{35}{5} = 7$

Exercises the students work depict these relationships, first in relation to the number line. (Figure 16.) For each whole number, students link the fraction and the rotated number family. Later, when students write the whole numbers for fractions, the students must rely on their knowledge of multiplication. (Figure 17.)

Figure 16.

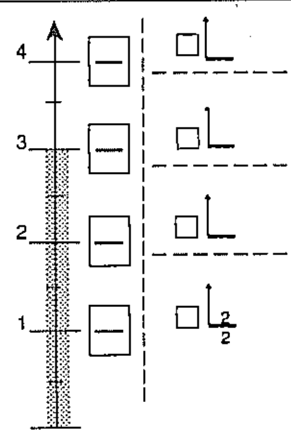


Figure 17.

a. $\square = \frac{20}{4}$ b. $\square = \frac{14}{7}$

At this point, students are shown a final relationship with fractions—division. The multiplication number family is returned to its original position.

$7 \,\big|\, \frac{14}{2}$ becomes $2\overline{)14}^{\,7}$ which is also $2\overline{)14}^{\,7}$

$5 \,\big|\, \frac{35}{7}$ becomes $7\overline{)35}^{\,5}$ which is also $7\overline{)35}^{\,5}$

Thus, students are shown that

$\frac{35}{7} = \square$ is the same as $7\overline{)35}^{\,\square}$.

Students work exercises that involve completing the related division problem and fraction. (Figure 18.)

In summary, rather than learning many different notions of fractions, students learn the relationship between fractions and whole numbers. Both multiplication and division come into play in demonstrating these relationships.

Figure 18.

a.	$5\overline{)40}$	$\frac{\square}{\square} =$
b.	$\overline{)\quad}$	$\frac{18}{3} =$

Division is then linked to the place value system represented by coins. (Figure 19). Children work problems with divisors of 5, representing nickels. Remainders are represented with pennies. Students determine the number of nickels, the quotient, and the number of pennies, the remainder, for each problem.

The third-grade level of *Connecting Math Concepts* was designed and field tested to develop both understanding and proficiency in mathematics. The strategies described in this article are taught systematically over many lessons. The learning that can result from this program not only is valuable in its own right but also sets the stage for far more sophistacted problem solving in later grades, which is described in the next article. ◆

Figure 19.

a. $5\overline{)42}$ $\overline{)\quad}$ P b. $5\overline{)59}$ $\overline{)\quad}$ P c. $5\overline{)21}$ $\overline{)\quad}$ P

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Teaching Problem Solving in Mathematics

by Don Steely, Systems Impact Inc.
Doug Carnine
Siegfried Engelmann, University of Oregon

The National Council of Teachers of Mathematics claimed in 1980 that "problem solving must be the focus of school mathematics in the 1980's" (p.2) and again emphasized it in their 1989 *Standards*, "The development of each student's ability to solve problems is essential..." (p.5)

Despite the concern about problem solving over the last ten years, educators have not yet even agreed on its definition. Some have decided that problem solving involves written problems that "require students to read several sentences, decide how to organize the problem, and to solve or compute the problem they have created" (Wheeler & McNutt, p. 309, 1983). Some think that problem solving "should involve a child in gathering, organizing, and interpreting information so that he can use mathematical symbols to describe real world relationships" (Ashlock, Johnson, Wilson and Jones, p. 239, 1983). Still others see it as a "selected sequence of activities, situations, contexts, and so on, from which students will, it is hoped, construct a particular way of thinking" (Thompson, 1985, p.191).

Probably greater consensus could be reached on a general statement of what we want students to be able to do, and that would be something akin to "being able to select and use a wide range of strategies or strategy combinations to solve a wide range of problems that vary in complexity and type of information given." This general definition of problem solving encompasses six prominent areas of mathematical problem solving research as suggested by Kameenui and Griffin (1989)—cognition, metacognition, learner characteristics, word problem characteristics, word problem classification, and instructional techniques and programs—as well as the three general factors of student achievement as suggested by Porter (1989): learner aptitude, pedagogical practice, and opportunity to learn.

Cognition, Metacognition, and Learner Aptitude

Cognition, metacognition, and learner aptitude can be grossly categorized as internal learner variables. They are not directly observable and can only be inferred by judging how well the learner performs on specific tasks.

Although the following definitions of cognition and metacognition will surely draw some criticism, they are general enough to perhaps allow agreement on the overall dimensions. Cognition is multidimensional knowledge that allows students to understand and interpret the problem and then to arrive at a decision about what approach should be taken to solve it. Metacognition is that ability which allows students to link and organize knowledge in a way that facilitates solving complex, multifaceted problems (Prawat, 1989; Polya, 1973; Flavell, 1971; Bruner, 1960).

Regardless of exactly how cognition and metacognition are defined, they are acquired capacities. They are the result of what students learn, and what students learn is primarily a result of the direct and indirect instructional experiences the student receives. If students are not proficient problem solvers, the instructional experiences have not been effective, or those experiences have inadvertently taught inefficient or undesired strategies and linkages.

Of course learner aptitude affects a student's competence at solving mathematical problems, both in terms of cognition and metacognition. These characteristics include computational ability (Balow, 1964), verbal ability (Alexander, 1960), general intelligence (Aiken, 1971), and knowledge of underlying numerical and fundamental arithmetic concepts (Chase, 1960). Like cognition and metacognition, these learner characteristics are a result of what students have learned and are a product of students' instructional experiences, though genetic factors can certainly be assumed.

Inherent in every new experience with a strategy or combination of strategies is relevant background knowledge that the learner must apply to utilize the strategy. The skills and abilities the learner brings to this experience directly affect the learner's ability to acquire and link the new strategies. If the relevant background knowledge is not present, then the experience with the strategy requires the learner to leap beyond his/her level and learn the background knowledge and strategy simultaneously, which can easily result in confusion.

Regardless of why the learner did not attain certain characteristics or background knowledge, the lack of those abilities may seriously impact the development of the learner's cognition and metacognition, the learner's ability to master and link strategies. It is a variable of the "experience" that must be dealt with by the instructional program.

Pedagogical Practice-Instructional Techniques and Instructional Programs

Instructional techniques are the methods we use to convey the content of the program, the endless stream of "experiences." Some techniques may be effective only with certain types of content, while other techniques are useful across a wide range of content. While curricular programs and techniques can be analyzed independently of one another, they are sometimes so tightly woven that it is difficult to separate them. As a unit, they determine how well the students learn from the experiences. For example, we may have a very elegant program and still not produce competent problem solvers due to poor techniques, such as a lack of sufficient practice, an over-dependence on teacher dialogue, or a lack of sufficient structured and guided practice. On the other hand, we may have very good techniques coupled with a very poor program and reach the same outcome, poor problem solvers.

In reviewing three studies, Porter (1989) summarizes four weaknesses that directly affect the development of problem-solving skills. The first three of these weaknesses are associated with instructional techniques: (1) an inordinate amount of time is spent teaching computational skills, at the expense of concept understanding and problem solving (further corroborated by Perkins and Simmons (1988) and Hamann and Ashcraft (1986)), (2) 70% or more of the topics covered received less than 30 minutes of instruction time (these were "taught for exposure"), (3) large differences exist in the actual amount of time teachers spend teaching mathematics.

The fourth weakness, the "low-intensity curriculum," was also cited by the 1987 Second International Mathematics Study, which lays the major blame of poor student performance on the spiral curriculum. "Content and goals linger from year to year so that curricula are driven by still unmastered mathematics content begun years before." (p.9) As an example, the topic of fractions is introduced in the kindergarten level of a 1991 edition of one math series and continues through grade 8. According to the suggested pacing guidelines, by the end of 8th grade students will have spent more than 120 days on fractions, most of it reviewing and reteaching skills from previous years. The analysis of another currently popular math series shows that 76% of the material in grade 6, 80% in grade 7, and 82% in grade 8 is review. Despite this siginificant amount of instruction and review, other factors must be involved. Why can only one-third of seventh grade students add fractions such as 1/2 and 1/3 (Peck, 1981)?

A study currently underway at the University of Oregon has identified a number of these factors (Carnine, in press). Although content is reviewed from year to year, review is inadequate. Immediate and frequent review is rare, and in one program new material was reviewed an average of only once every 20 days.

A second factor has to do with the instructional strategies used in teaching problem solving. Those strategies often cover only a very limited range of problem types (such as a fraction followed by "of" means multiply), or are so general they do not generate any specific plan for dealing with any particular problem (such as "Read, Plan, Solve, Check").

A third factor is the rate at which new concepts are introduced. Although a great amount of time is spent on a particular topic in a textbook, most lessons present many new concepts simultaneously, often attempting to teach every variation and nuance. Sporadic review, vague strategies, and rapid rate of introduction of new concepts exacerbate the problems brought about by a low-intensity, spiral curriculum.

Opportunity to Learn—The Problems to be Solved

Although many educators do not consider word problems to be problem solving, word problems are the most frequent manifestation of problem solving in textbooks. Admittedly, word problems represent only one type of problem solving behavior; however, they are important and have been the subject of extensive research.

Problem characteristics that have been shown to be important variables in the difficulty of problems include *semantics* (DeCorte, Verschaffel, & DeWin, 1985; Riley Greeno, & Heller, 1983; Sandburg & De Ruiter, 1985), *syntax* (Larson, Parker, & Trenholme, 1978; Wheeler & McNutt, 1983; Moyer, Moyer, Sowder, & Threadgill-Sowder, 1984; Greeno, 1980, Heller & Greeno, 1978) and the *presence of extraneous information* (Arter & Clinton, 1974; Carpenter, Hiebert, & Moser, 1981; Cohen & Stover, 1981). Other characteristics that have been researched include *question placement* (Arter & Clinton, 1974), *order of numerical data* (Burns & Yonally, 1964; Cohen & Stover, 1981; Jerman & Rees, 1972; Rosenthal & Resnick, 1974), *presence of tables, charts, or pictures* (Moyer et al., 1984)), and *number of steps required to solve the problem* (Loftus & Suppes, 1972; Mayer, 1982; Quintero, 1983).

Two general types of classification systems for problems have been proposed: (1) learner-derived strategies and (2) task-driven requirements of the problems. If it is accepted that learners develop problem-solving strategies based on instructional experiences, then learner derived strategies are circuitous. In any program, problems are not presented entirely at random. They are presented in an order based on certain suppositions about useful strategies. The learner learns those strategies, as well as others that are unintended. The instructional experiences associated with those problems either foster or hinder the development of competent problem solving. Certainly it is interesting to see how the set of instructional experiences has been interpreted by the learner. However, the learner's interpretation cannot then be used to *justify* a learner-dervied classification system that then becomes the basis for selecting an order for presenting problems.

The other tactic for classifying problems is based on the requirements of the problems themselves. These analyses have focused on *actions* (Underhill, 1977), degrees of abstraction and factualness (Caldwell and Goldin, 1979), hierarchical task analysis (Silbert, Carnine & Stein, 1991), and conceptual groupings (Peterson, Fennema, & Carpenter, 1989).

Instructional programs are the sequences of experiences (problems of different categories with different characteristics) presented to the learner. Which problem characteristics are dealt with, and how problems are classified define the student's opportunity to learn problem solving. This is an often overlooked determinant of opportunity to learn.

Summary (Part 1)

Although there are many facets of problem solving ability, most, if not all, depend on prior instruction. Cognition and metacognition are the results of the student's experiences. Likewise, learner abilities are a function of what has been previously experienced and mastered. Instructional techniques, while crucial to the effective transmittal of these experiences, is distinct from the actual curriculum-based experiences. This leaves the curricular material, the experiences with solving problems, as the starting point, the cornerstone.

Any curricular program consists of much more than just problem solving. In fact, non-problem-solving parts of the program actually determine what types of problem solving experiences can be reasonably attempted. However, it is the problem solving component that has received the most interest, because it represents higher order thinking, is more difficult to teach, and is generally less effectively taught.

The development of an effective instructional program for teaching problem solving skills requires a classification of problems that accounts for the entire range of problem types and relevant characteristics. The program must also insure that the learner abilities include all of the relevant and necessary background knowledge (non-problem-solving abilities). If these experiences are developed well, and if they are combined with good teaching techniques, then there is a reasonable chance of furthering the learner's cognition and metacognition.

The system for classifying problems and dealing with the wide range of problem characteristics defines the strategies presented in a program. These strategies can be numerous, each dealing with a small range of problems, or can be fewer and more general, based on important samenesses shared by all the problem types. The effectiveness of these strategies directly affects the student's ability to use any one particular strategy and to link strategies. The importance of identifying samenesses for developing these strategies and their inter-relationships is the subject of the remainder of this paper.

Problem Solving in Grades 1 though 4

In the primary grades, poor problem solving skills arise primarily from an inability, or possible reluctance, of mathematics educators to devise explicit strategies that students can learn and successfully apply. The consequence is the avoidance of any but the most rudimentary type of addition and subtraction word problems in primary textbooks (Peterson, Fennema & Carpenter, 1989).

The following analysis, taken from *Connecting Math Concepts* (Engelmann & Carnine, 1991), teaches students to see the relationship between the situation described in a story problem and the concept of a number family composed of two small numbers and a big number. As noted in the previous article, number families are first used to teach basic addition and subtraction facts. The number family structure prompts important relationships between addition and subtraction and reduces the number of sets to be memorized from 200 to 55.

Teaching the fact number families also promotes the integration of the concepts of addition and subtraction. When a small number is missing in a number family, (e.g. $\underline{\quad}\ \xrightarrow{\quad 9\quad} 17$), the students learn to start with the big number and subtract the smaller number shown: 17- 9=__. Conversely, if the big number is missing, e.g. $8 \xrightarrow{\quad 9\quad} \underline{\quad}$, the students add the two small numbers to determine the big number: 8 + 9 = __ .

Number families provide a "map" that can be used to diagram the various types of word problems. The number family map in turn leads to setting up the addition or subtraction calculation. The strategy teaches students not to make quick judgments because of the presence of a "key" word such as *more.* First, students graphically represent the situation described in the word problem. Second, they determine how to write the number problem.

In the word problem below about Mark, at-risk students are likely to add 66 and 121, because the problem says that Mark will gather more nuts. However, adding 66 and 121 does not lead to the correct answer.

> Mark gathered some nuts before lunch. After lunch he gathered 66 more pounds of nuts. At the end of the day he had 121 pounds of nuts. How many pounds of nuts did he gather before lunch?

This problem tells about getting more. So the students write the small numbers first and end with the big number, the pounds of nuts Mark had at the end of the day. As students have learned from working with number families, the two small values go on top of the arrow and the big number goes at the end: $\underline{\quad\ \ 66\ \ }\!\!\rightarrow 121$

The students then apply what they have learned about the relationships between addition and subtraction families to compute the answer. When the unknown in a number family is a small number, as in this example, the number family can be translated into a subtraction problem (121 – 66 =__) to produce the other "small number": 121 – 66 = 55.

In the following comparison problem, many students will subtract because of the words "weighed less"

> Mark weighed 46 pounds less than Lois. Mark weighed 102 pounds. How much did Lois weigh?

The sentence that tells about the comparison, "Mark weighed less than Lois," tells that Mark must be represented by a small number. That same sentence indicates that Mark weighed 46 less, so 46 is the other small number. By default, Lois is the big number.

$\underline{46 \quad M} \rightarrow L$

The problem also gives a number for Mark, so Mark can be replaced with 102.

$\underline{46 \quad \overset{102}{\cancel{M}}} \rightarrow L$

Because the problem states both small numbers, the students write an addition problem, 46 + 102=__, which tells how much Lois weighed.

Classification problems can use the same heuristic. To work these problems, students need to understand the relationships among the classes named in a problem. For example, hammers and saws are members of the bigger class, tools; or, as in the following problem, chairs and couches are members of the larger class, furniture.

> A hotel is going to buy 112 pieces of furniture. It needs to buy 57 couches. The hotel will buy chairs for the rest of the furniture. How many chairs will the hotel buy?

Furniture is the "big" class, and the names for the subordinate classes go in the places for the "small" numbers.

$\underline{\text{couches chairs}} \rightarrow \text{furniture}$

The problem gives one of the small numbers, couches, and the large number, furniture.

$\underline{\overset{57}{\cancel{\text{couches}}}\ \text{chairs}} \rightarrow \overset{112}{\cancel{\text{furniture}}}$

The students know the big number, so they subtract: 112 - 57 = __. The answer gives the number for chairs.

This same number family heuristic also works for multi-step addition/subtraction problems that involve a person having some money, spending some money and then ending up with some money. In these problems, the money a person has is always the big number.

> Fran had \$36. Then her mother paid her \$12 for work around the house. Fran wants to buy a blouse, but she needs to keep \$14 for a bike light. How much can Fran spend on a blouse?

Two numbers must be added to find how much Fran has.

$\underline{\text{ends with} \quad \text{can spend}} \rightarrow \underset{\cancel{\text{has}}}{\$36 + \$12}$

The amount she must end with is substituted,

$\underline{\overset{\$14}{\cancel{\text{ends with}}} \quad \text{can spend}} \rightarrow \underset{\cancel{\text{has}}}{\overset{\$48}{\cancel{\$36 + \$12}}}$

The resulting problem, 48-14=__, gives the amount Fran can spend on a blouse.

This same type of analysis applies when multiple amounts are spent:

> Fran had \$42. She spent \$18 on running shoes. She spent \$4.20 on school supplies. How much did she end up with?

$\underline{\text{ends with} \quad \overset{\overset{\$22.20}{\cancel{\$18 + \$4.20}}}{\cancel{\text{spend}}}} \rightarrow \underset{\cancel{\text{has}}}{\$42}$

and when multiple amounts are earned and multiple amounts are spent:

> Ginger had \$42. She earned \$12 at her job. She went to the store and bought a swimsuit for \$25 and swim fins for \$16. Ginger gave her brother the \$6 that she owed him. How much did she end up with?

$\underline{\text{ends with} \quad \overset{\overset{47}{\cancel{25 + 16 + 6}}}{\cancel{\text{spend}}}} \rightarrow \underset{\cancel{\text{has}}}{\overset{54}{\cancel{42 + 12}}}$

In the following problem type, students must identify the relevant information from a table before they can add and then subtract.

Earn		Spend	
Mow the lawn	\$ 5.00	Goldfish bowl	8.00
Clean the garage	7.00	Running shoes	28.00
Babysit	8.00	Skateboard	45.00
Sell your bike	40.00	Movie ticket	6.00
Win a prize	9.00	Music tape	11.00
Receive allowance	12.00	Sweatshirt	21.00
		Gloves	18.00
		Socks	4.00

Henry has to earn money to buy some things he wants. He mows the lawn and sells his bike. With the money he makes, Henry buys a tape and running shoes. How much does he end up with?

39
~~11 + 28~~
ends with ~~spend~~ ⟶ 45 ~~5 + 40~~ ~~has~~

The problem 45-39= __ gives the amount he ended with.

This number family heuristic will continue to provide a concrete strategy when students incorporate multiplication and division. Key words, such as *each, every,* and *equal parts,* cue the students that multiplication or division is involved, and not addition or subtraction.

To differentiate multiplication/division number families from addition/subtraction number families, students write the families this way:

$$2 \overset{9}{\longrightarrow} 18$$

The big number is still at the end of the arrow. Also, 55 families yield all 200 facts. For the 2-9-18 family, the facts are $2 \times 9 = 18, 9 \times 2 = 18, 18 \div 2 = 9$, and $18 \div 9 = 2$.

The number family analysis also applies to multiplication and division word problems.

If each shirt requires 2 yards of material, how much material will be needed to make 5 shirts?

This problem tells about yards of material per shirt, or:

$$\frac{\text{yards}}{\text{shirt}}$$

which can be written as yards ÷ shirts or shirt yards. The quotient expresses shirts per yard and thus the number family becomes

$$\text{shirts} \xrightarrow{\frac{\text{yards}}{\text{shirt}}} \text{yards}$$

The students put in the values they know

$$5\ \cancel{\text{shirts}} \xrightarrow{\frac{2\ \text{yar}\cancel{\text{ds}}}{\cancel{\text{shirt}}}} \text{yards}$$

and then select the operation. The two small numbers are given, so the students multiply to determine the big number, 5 x 2 = __. Five shirts would require 10 yards of material.

The number family strategy provides students with a explicit means of dealing with a wide range of problem classifications and problem wordings across all four basic operations. There is a *sameness* in all of these problems that is emphasized by the strategy, a sameness which allows students to use one strategy to analyze and solve all the various problem types.

Problem Solving in Grades 5 through 7

The analysis of ratio/proportion problems in grades 5-7 provides another example of the economy of teaching samenesses. Typical instruction involves showing the students how to set up a ratio for a particular type of problem, e.g., using map scales, and having the students solve similar map scale problems in which the missing part of the ratio is either the top or bottom of the second fraction. *On this map, one inch represents 5 miles. How long would a line be for a distance of 14 miles?* Depending on the instruction of a typical program, students may only see one or two other treatments of ratios in lessons that appear several months later.

However, a thorough sameness analysis of ratio problems (Systems Impact, 1988) would include many more types of problems. In the first group of ratio problems, students set up the labels for a ratio based on repeated unit names.

Five packages make 4 gallons. Seven Packages would make how many gallons?

Packages	Gallons
$\frac{5}{7}$	$\frac{4}{n}$

They then fill in the numbers to complete the ratio and solve it.

Given this analysis, students can set up ratios in which:

- the unknown is in the first fraction:

If a scale drawing shows that 5 cm equals 12 km, how long is a line that shows 15 km?

Centimeters	Kilometers
$\frac{5}{n}$	$\frac{12}{15}$

- the question comes first:

How many cents will 7 packages cost if it costs 45¢ for 5 packages?

Cents	Packages
$\frac{N}{45}$	$\frac{7}{5}$

- the order of the data is inverted:

If a car travels 200 miles in 6 hours, in 8 hours how many miles could it travel?

Miles	Hours
$\frac{200}{N}$	$\frac{6}{8}$

- extraneous information is given:

Sam is making punch for 15 friends. If 5 packages of mix make 4 gallons of punch, how many packages will he need to make 7 gallons of punch?

Packages	Gallons
$\frac{5}{n}$	$\frac{4}{7}$

- the information is given graphically or in a table

Brownies—makes 15
2 cups flour
1/2 cup butter

Brad has a recipe for making 15 brownies but he needs to make 24 brownies. How much butter does he need?

This analysis also permits the inclusion of percent problems. Percent problems are normally taught as three entirely different problem types, depending upon which parts are known. If both numbers are known, the strategy is usually to divide: The *team played 50 games and won 44. What percent of their games did they win?* This problem translates into $44 \div 50 = \square$. If the problem tells a percent and gives the total number, the strategy is to multiply: *The team played 50 games. They won 88% of their games* ($50 \times .88 = \square$). If the problem tells the percent and a part of the total, the strategy usually requires an algebraic solution: *The team won 88% of their games. They won 44 games. How many games did they play?* ($.88x = 44$). However, all of these types can be handled through the same ratio strategy.

The team played 50 games and won 44. What percent of their games did they win?

Percent	Games
$\frac{n}{100}$	$\frac{44}{50}$

The team played 50 games. They won 88% of their games. How many games did they win?

Percent	Games
$\frac{88}{100}$	$\frac{n}{50}$

The team won 44 games. They won 88% of their games. How many games did they play?

Percent	Games
$\frac{88}{100}$	$\frac{44}{n}$

By teaching the common convention that all of a number is 100%, students can use the same analysis to solve many other types of percent problems.

	Number	Percent
• *Eight is 52% of what number?*		
"Eight is 52%" translates into this row	<u>8</u>	<u>52</u>
"Of what number" translates into this row	n	100
• *70 is what percent of 25?*		
"70 is what percent" translates into this row	<u>70</u>	<u>n</u>
"of 25" translates into this row	25	100
• *What number is 42% of 30?*		
"What number is 42%" translates into this row	<u>n</u>	<u>42</u>
"of 30" translates into this row	30	100

By learning the common convention that "per hour" means "per one hour", students can use the same analysis to solve rate problems. Rate problems are normally taught with another strategy that requires memorizing the formula D=RT, or in some programs, memorizing all three variants of the formula.

	Miles	Hours
• *If a plane travels 650 miles per hour, how many miles will it travel in 3 hours?*	$\frac{650}{n}$	$\frac{1}{3}$
• *What is the average rate of a car that goes 450 miles in 9.5 hours?*	$\frac{n}{450}$	$\frac{1}{9.5}$
• *How long will it take the train to go 540 miles to Rome if it travels at 120 mph?*	$\frac{540}{120}$	$\frac{n}{1}$

This analysis thus far has included problems that give just two references to the same units, e.g. *miles* and *hours* are mentioned (or directly implied) twice in the problem. However, this same analysis can be extended to more elaborate problems in which one of the "unit" names is a classification. These problems require setting up a table, but the same basic structure applies.

120 skiers can go on a ski trip. Forty percent must be good skiers. The rest can be beginners. How many beginners can go?

The units are skiers and percents. The class of people include good, beginners, and total people.

	Skiers	Percent
good		
beginners		
total		

Numbers are given for total skiers and percent good skiers. The total is always 100%, so that number can be filled in. The percent for beginners can be calculated by subtracting 40% from 100%.

	Skiers	Percent
good		40
beginners		(60)
total	120	(100)

A ratio can now be written using the row with the missing number for beginners and the row with two values.

Ratio

$$\frac{n}{120} = \frac{60}{100}$$

The same analysis and structure with the table applies to a wide range of problems. For example:

75% of the mixture must be sand. The rest is gravel. 800 pounds of the mixture is needed. How many pounds of gravel must be used?

	Pounds	Percent	Ratio
sand		75	$\frac{n}{800} = \frac{25}{100}$
gravel		(25)	
total	800	(100)	

The discount on an item is 25%. The regular price is $14.50. How much is the sale price?

	Dollars	Percent	Ratio
sale price		(75)	$\frac{n}{14.50} = \frac{75}{100}$
discount		25	
total	14.50	(100)	

A mixture contains peanuts and almonds in a ratio of 5 to 1. If 7 kilograms of almonds are used, how many kilograms of mixture is made? How many kilograms of peanuts are used?

	Kilograms	Ratio	Ratio
peanuts		5	$\frac{7}{n} = \frac{1}{6}$
almonds	7	1	
total		(6)	

A truck is going to haul 10 canoes, 8 rowboats, and some rafts. If the truck can hold 30 boats, what percent will be rafts?

	Boats	Percent	Ratio
canoes	10		$\frac{12}{30} = \frac{n}{100}$
rowboats	8		
rafts	(12)		
total	30	100	

The dealer's cost for an item is $5.70. If he marks up the item 45%, what is the marked-up price?

	Boats	Percent	Ratio
cost	5.70	100	$\frac{5.70}{n} = \frac{100}{145}$
markup		45	
marked-up price		(145)	

John worked for 42 minutes mowing, trimming, and sweeping. If he mowed grass 1/2 of the time and trimmed for 1/3 of the time, how long did he spend sweeping?

	Minutes	Fraction	Ratio
mow		1/2	$\frac{n}{42} = \frac{1/6}{1}$
trim		1/3	
sweep		(1/6)	
total	42	1	

The list of possible problems can be extended to include more problem types involving won-lost percents, percent increases and decreases, problems that require students to change one unit of measure to match another, averages, and probability.

This sameness analysis takes a single basic skill of setting up a ratio based on repeated unit names, explains what 100% means, demonstrates how to put data into a summary table, and ends up covering literally hundreds of different problem types that are normally taught as unrelated problems.

A General Paradigm for Problem Solving

These examples illustrate how the sameness analysis can lead to a more coherent, elegant, and linked understanding of word problems, rather than fragmented knowledge and strategies. The overriding purpose of sameness analysis is to foster coherent schemas. Although this goal implies great ambitions, how does it apply to problems beyond those involving number families, equations, and ratios; i.e., what is the broader scope of sameness analysis as it applies to problem solving in mathematics?

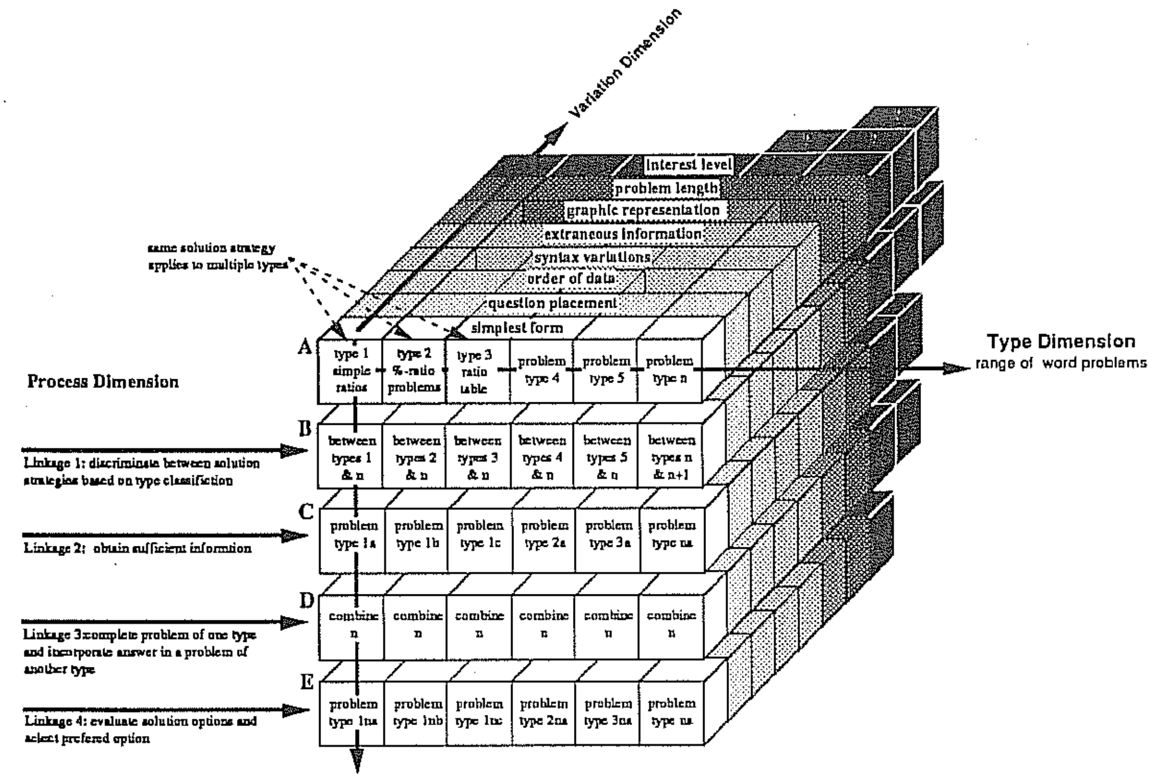
The Type-Process Matrix (Figure 1) attempts to assimilate both an analytical type categorization of problems as well as a process analysis for developing higher-order thinking skills. There are three basic dimensions that are applicable to our analysis of sameness, represented in Figure 1 by the arrows: (1) the type dimension, (2) the variation dimension, and (3) the process dimension. Although there are significant interactions between these levels, we'll deal with them one at a time.

The type dimension is the starting point that lists the concepts and problems that are being considered. This dimension will change as we include examples of more problems across more grade levels and as the students become more sophisticated. For the purpose of this article, assume that the range to be considered consists of word problems typically found in secondary math texts.

The sameness analysis across this dimension asks, "When all of the concepts and problems within this universe are considered, *what similar solution strategies appear?*" Given our universe of word problems, we would classify the *ratio/proportion* solution strategy that applies to ratio, percent, rate, and probability as one type.

Another set of problems that has a similar solution strategy, one quite different from ratios, includes a wide range of problems that deal with trigonometric ratios (sine, cosine, tangent). This common solution strategy represents a second type.

Figure 1. Type-Process Analysis Matrix for Secondary Word Problems



A third type of word problem translates into two-step algebraic equations, e.g., age, coin, and mixture problems that deal with percent concentrates or express the value of one unknown in terms of the other. Again, these problems can all be solved with a common solution strategy.

The result of a thorough sameness analysis across the *type* dimension gives us *types of problems* grouped according to *common solution strategies*. The number of types depends on how great our range of examples is and how well we can determine samenesses in problems that may appear quite different. There may be problems that could be classified as belonging to a particular type but are so different in so many respects that students would find it difficult to recognize the sameness. These problems are usually excluded. Practical concerns such as these illustrate that the matrix is not intended to be exhaustive for theoretical purposes but practical for guiding the design of curriculum.

The *variation dimension* deals with all possible variations within a single type of problem. Depending upon how we have defined our type dimension, these variations may be seemingly infinite. But for the sake of simplicity, let's continue with our analysis of word problems, starting with a basic ratio problem type (e.g., "If 5 packages make 4 gallons, 6 packages make how many gallons?"). What are the possible variations that apply to this problem and might also apply to other such word problems? We could:

- put the question first or in the middle
- change the order of the numbers
- change the syntax of the problem
- give information that is not needed
- make the problem more "real life"
- change the perceived interest level of the problem

These, and others, would not only be variations of this single problem type, but also variations for other ratio/proportion types and other non ratio/proportion types. There are probably other variations as well as combinations of variations.

The goal in analyzing the variation dimension is to expand the range of application through the transfer of a strategy to new problem variations. For learners to fully understand and appreciate the power of a solution strategy, they must see that the solution strategy applies across all variations of the type. Despite the outward appearance, problems within a type are the same in terms of the solution strategy:

- Putting the question at the front of a problem doesn't change the structure.
- The order of data in a problem is sometimes mixed up, but that doesn't change the structure.

- Extraneous information in a problem is irrelevant and doesn't change the structure.

These generalizations about problem structure are also important when other solution strategies are taught; less time will be required to show the solution strategy sameness across the variations.

The *process dimension* comes into play when, at a minimum, one particular solution strategy must be differentiated from other strategies. On the simplest level (linkage 1 in Figure 1), analysis of the process dimension must look at problem types that have similar solution strategies and determine how students can tell which solution strategy to use. For example, how do students know whether or not to use a table for a ratio problem (i.e.how do students know if the ratio problem is a type 1, 2 or 3?).

On a slightly more complex level (still linkage 1 in Figure 1), the analysis must look at all problems in a solution strategy and determine what they have in common that allows the students to determine when to use that solution strategy as opposed to quite a different solution strategy. Certainly this is a direct reflection of samenesses that led to similar solution groups when the type dimension was first analyzed. As an example, "What determines if the following problems can be solved with a ratio strategy or with a different strategy that involves a two-step algebraic equation?

> *A mix contains peanuts and almonds in a ratio of 4 to 3. If 36 pounds of mix are made, how many pounds of almonds will be used?*
>
> *A mix contains peanuts, costing $2.00 per pound, and almonds, costing $5.00 per pound. Thirty-six pounds of mix are made and it sells for $4.00 a pound. How many pounds of peanuts are used?*

Both problems seem to fit a ratio setup; the first gives information about pounds.

	Pounds	Ratio
peanuts		4
almonds	n	3
total	36	(7)

Ratio

$$\frac{n}{36} = \frac{3}{7}$$

The second problem, however, gives information not about dollars, but about dollars per pound. The second problem requires a table from which an algebraic equation can be derived:

	Pounds	$/Pound	Dollars
peanuts	[x]	2.00	[2x]
almonds	[36-x]	5.00	[5 (36-x)]
total	36	4.00	[2x + 5 (36-x)]

Obviously this process dimension is in a constant state of flux, as new solution strategies are continually being learned and linked.

On a more complex process level (linkage 2 in Figure 1), learners must know all *the attributes of problems* that can be solved with a particular solution strategy. This becomes imperative when a problem is incomplete, or when one of the requisite pieces of information must be calculated before it fits the standard problem type. For example, *a ratio problem requires 2 unit names and numbers for 3 of the 4 values* in the ratio before it can be solved. The following problem requires the addition of numbers before a recognizable problem type emerges.

> *Sam used 2 cups of sugar to make 2 quarts of punch. It wasn't sweet enough, so he added another cup of sugar, and then one more to make the punch sweet enough. If Sam needs to make 5 quarts of punch, how much sugar will he need?*

Cups Quarts

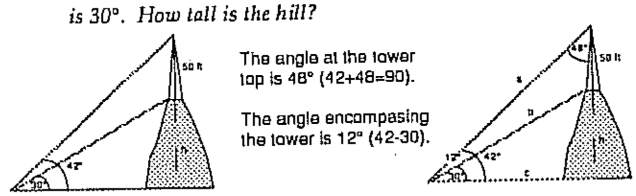
$$\frac{2+1+1}{n} = \frac{2}{5}$$

The sugar and punch example actually requires two separate solution strategies—adding and solving a ratio. However, at the secondary level, a simple addition problem is probably too elementary to be included in our range of problem types. It may be considered an "internalized learner process." Over time, more and more of the solution strategies become "internalized" or automatic.

Linkage 2 in Figure 1 also includes gathering data so as to have sufficient information to solve the problem. For example, students could actually taste three samples of punch made with two, three, and four cups of sugar. The sample preferred by the most students would determine the number of cups of sugar to put in the ratio.

On an even more complex process level (linkage 3 in Figure 1), learners must apply this sameness analysis to problems that may truly combine two different, non-internalized, solution strategies.

> *A 50 foot tower sits on a hill. John's line of sight to the top of the tower is 42°. His line of sight to the bottom of the tower is 30°. How tall is the hill?*



Two solution strategies must be used for this problem. First students must recognize that they can find the hill height by using the standard sine ratio,

$$\sin 30° = \frac{h}{b}$$

But to use this ratio, they must first find the value of b, which they can do by using the law of sines,

$$\frac{b}{\sin 48^\circ} = \frac{50}{\sin 12^\circ}$$

In this example, using the law of sines to determine the line-of-sight distance to the bottom of the tower and then using the sine ratio to solve for the hill height are closely related strategies. In other problems the two solution strategies may be more different.

It is conceivable that every solution strategy could be combined with every other strategy. Since each strategy may apply to several problem types, we are left with an enormous number of combination problems. Just consider this last example—we could word the problem so that after using the law of sines,

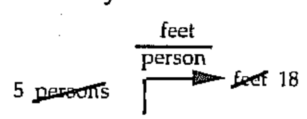
- the height must be figured by a ratio
- the height must be figured by using an algebraic equation
- the height must be figured by using the law of sines again

Due to time constraints, it would be impossible to teach all of the possible combinations of all solution strategies and problem types. This fact emphasizes the importance of insuring that the learner has a complete understanding of the characteristics of each solution strategy and the similar problem types that strategy will work for. Students might then have acquired the metacognition to monitor the selection of strategies and strategy combinations that might be required for a particular problem.

The process described by linkage 4 in Figure 1 gives students the option of picking from 2 or more appropriate solution strategies. Although a problem type may have been taught using a particular solution strategy, other possible strategies will often be possible. As students learn these new strategies, they may realize that a problem can be solved in different ways. As a simple example, consider the following problem:

> *A piece of rope 18 feet long must be divided into 5 equal parts. How long a piece will each person receive?*

This problem would most likely have been taught using a number family solution:



The big number is given, so students must use division to find the other small number—$18 \div 5 = 3.6$

This problem could also be solved as a ratio problem:

Feet	Persons
$\frac{18}{n}$	$\frac{5}{1}$

In more complicated problems, students may be able to solve parts of the problem in different ways. This type of linkage requires firm mastery of previous linkage levels. Although it is enlightening to show students their options, instructional time is often so limited that this last type of linkage is left primarily to incidental learning.

Summary (Part 2)

Within the type dimension in Figure 1, the analysis shows which problem types can be solved with the same strategy. This will give guidance to the order in which to present different problem types. Instead of different types of problems being fragmented, they can be more economically taught as a sequential group. The learner benefits from more rapid assimilation and more comprehensive understanding of how the types are related.

Within the variation dimension, the analysis shows common variations across different problem types and groups of problem types. Once a particular variation has been shown across several members of that group, transfer to other members of the group occurs. It is not necessary to show all variations with all types.

Within the process dimension, the analysis reveals the characteristics of similar solution groups that must be shown for the learner to differentiate between when to use different solution strategies, when to alter a problem to fit a solution strategy, when to combine solution strategies, and how to evaluate solution strategy choices.

The type-process matrix provides a means of identifying important samenesses and linkages within a specified set of problem solving experiences. If we want students to be able to select and use a wide range of problem solving strategies and strategy combinations across a wide range of problems, then we must provide them with experiences that effectively teach those explicit strategies. To be useful, the students must see that the strategies are generalizable across a wide range of problems and must understand all of the components of problems that can utilize a particular strategy.

The analysis we have presented is complex because problem solving in mathematics is complex. Some students, who are facile problem solvers to begin with, figure out on their own how to solve these types of difficult problems. Our concern, however, is with those atypical learners who require explicit instruction to be able to develop sophisticated understandings. The research summarized earlier suggests that this is an obtainable goal. Ironically, explicit guidance in building schemas for complex content would probably benefit the vast majority of students, not just atypical learners. ◆

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The Mathematics Curriculum—Standards, Textbooks, and Pedagogy: *A Case Study of Fifth Grade Division*

by Jerry Silbert
Doug Carnine
University of Oregon

The purpose of this article is to compare the teaching of *division* in two fifth-grade basals published prior to the new National Council of Teachers of Mathematics (NCTM) *Standards* (1989) and the new editions of those basals that were published after the *Standards*. The focus is not on the *Standards* per se, which are clearly better represented in the 1991 editions, but on pedagogy as it relates to lower-performing students.

In mathematics education, the question is not "whether to reform," but "where to begin." One place is the curriculum, represented by math textbooks. As Farr, Tulley, and Powell (1987) noted, "Textbooks dominate instruction in elementary and secondary schools" (p. 59). This dominance has led to a closer look at the quality of textbooks. For example, Osborn, Jones, and Stein (1985) argue that "improving textbook programs used in American schools is an essential step toward improving American schooling" (p. 10). Such improvement is particularly important for special education and at-risk students.

State and local textbook adoption committees have the responsibility for evaluating the quality of textbooks and how well they might accommodate individual differences. Unfortunately, state adoption committees, in an effort to provide a standardized state curriculum, have in fact produced a "uniformity of curriculum" (Tulley and Farr, 1985). Because publishing companies must respond to the guidelines of large "adoption states," almost all textbooks are very similar. This homogeneity reflects the influence of the size of the state-wide adoptions in California, Texas, and Florida. To conform to the guidelines in these states, publishers end up acting as if all students "require or benefit from the same instructional goals and sequences," but to the extent that curriculum uniformity is achieved, "the ability to meet the diverse needs of students is reduced" (Tulley & Farr, 1985, p. 1). This observation is confirmed by the finding that pedagogy and educational research are seldom mentioned as factors influencing the judgments of selection committee members (Courtland et al., 1983; Powell, 1986).

The inattention to pedagogy and research findings contributes directly to ineffectual textbooks and indirectly to the low levels of student achievement in mathematics. For example, a relatively large percentage of the topics taught in mathematics receive brief coverage (Porter, 1989). Porter's finding about use of time is one of the many pedagogical variables identified in a research review by Dixon (1990). Other pedagogical variables suggest a number of criteria for evaluating math textbooks: What provisions are made to ensure that the students have the *relevant prior knowledge?* Is the *rate for introducing new concepts* reasonable? Is there a *logical coherence* in the presentation of strategies? Do the instructional activities *communicate in a clear, concise manner?* Is there an *adequate transition*, in the form of guided practice, between the initial-teaching stage and the stage where students work independently? Is *adequate review* provided to ensure that students will remember what they've learned?

In 1991, nearly all publishers of basal math programs are releasing new editions of their math pro-

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grams. These new editions reflect the publishers' reaction to the recent furor in math education, crystalized by the new NCTM *Standards*. However, in this article we do not describe the extent to which basal programs conform to the new NCTM *Standards*. Rather, we examined the programs according to the curriculum-design variables mentioned above that influence the learning of a broad spectrum of students.

Specifically, we evaluated basal mathematics instruction in two editions from the late '80s that did not have time to digest the new NCTM *Standards* and two editions from the '90s that reflect the new *Standards*. Only one publisher had its late 1980's math basal approved in the three largest adoption states—California, Texas, and Florida. The second late 1980's basal was selected because it more nearly exemplified the criteria for instructional effectiveness identified by Dixon (1990). This designation did not result from just evaluating division, but all the major topics in the text. Because of the basic similarity of all math basals, the actual differences in pedagogy are slight. A better description would probably be "slightly stronger pedagogy." Because our review of the basal texts confirmed earlier observations about textbook homogeneity, we will not single out the two publishers, but refer to them as the *higher-approval* and *stronger-pedagogy basals*.

We decided to study two-digit divisor problems in this paper because that skill requires students to utilize a wide number of concepts, and is usually difficult for teachers to teach and for students to learn. First, we compared the teaching of two-digit divisor problems in the 1988 edition of the *higher-approval basal* and the 1989 edition of the *stronger-pedagogy basal*. Then, we examined the corresponding chapters in the 1991 editions. The difficulty of teaching this algorithm explains, in part, why the new NCTM *Standards* deemphasize two-digit divisor problems. Using a calculator makes life easier for teachers and for the students. However, the '90s textbooks still teach the algorithm.

Pedagogical Criteria

Prior Knowledge

Prior to working two-digit divisor problems, students should have learned their division facts, subtraction with regrouping, rounding, and how to multiply a two-digit number by a one-digit number. Students must learn to carry out this multiplication in the new configuration shown below.

$$\begin{array}{r} 7 \\ 42\overline{)289} \end{array}$$

Both editions of the *higher-approval basal* and the *stronger-pedagogy basal* provide abbreviated teaching on division facts, rounding, regrouping, and multiplication at the fifth-grade level. Each skill is taught for just *one lesson*. The assumption seems to be that the students mastered these skills in earlier grades, and at this level they needed only a quick review. In terms of the expectations concerning prior knowledge, the '80s and '90s editions are quite similar.

Students who had not mastered the assumed skills previously would be in jeopardy because of the speed at which the new skills are presented. For example, the lesson that teaches rounding in the '91 version of the *stronger-pedagogy basal* began with a teacher model of how to round a 5-digit number to the nearest thousand, then provided a practice set in which the students rounded tens numbers to the nearest ten, hundreds numbers to the nearest hundred, thousands numbers to the nearest thousand, and ten thousands numbers to the nearest hundred (e.g., 10,564 rounds to 10,600).

Because the programs assume that students can learn the skills quickly, the success of the students on 2-digit divisor problems will depend to a great extent on how the teacher monitors the performance of the students on multiplication, rounding , and subtraction, and provides any necessary remedies.

Rate for Introducing New Concepts

When students work a problem, they first round off and then estimate. The estimated quotient may be too great or too small. The student must learn that an estimated quotient is too great if it is not possible to subtract after multiplying. For example:

$$\begin{array}{r} 9 \\ 42\overline{)361} \\ \underline{-378} \end{array}$$

When the estimated quotient is too great, the student must try a quotient that is less.

The student must learn that an estimated quotient is too small if the remainder is greater than the divisor. For example:

$$\begin{array}{r} 4 \\ 18\overline{)89} \\ \underline{72} \\ 19 \end{array}$$

The remainder must be less than the divisor. When the estimated quotient is too small, the student must try a quotient that is more. Note how potentially confusing this type of problem is. Without careful explanations, students are likely to become confused.

The evaluation of the rate of introduction for 2-digit divisor problems is based on an analysis of the various types of problems related to estimated quotients that are too large or too small. The four major types of problems are:

- 1-digit answer in which the estimating procedure produced the correct quotient.
- 1-digit answer in which the estimating procedure produced an incorrect quotient.
- 2-digit answer in which the estimating procedure produced a correct quotient.
- 2-digit answer in which the estimating procedure produced an incorrect quotient.

The rate of introduction of skills can be inferred from Figure 1, which shows the objectives from the chapter on 2-digit divisors in the '89 and '91 versions of the *stronger-pedagogy basal*, and the objectives from the '88 and '91 versions of the *higher-approval basal.*

Both '80s editions introduced the four types of problems at too fast a rate for average and lower-performing students. Unfortunately, these lessons remained fundamentally unchanged in the 1991 editions. The most critical aspect of two-digit divisor problems has to do with when the estimated quotients are *incorrect.* In the 1991 version of the *higher-approval basal,* two lessons (4 and 5) teach students how to solve problems with a one-digit quotient that is not the same as the estimated quotient, and only one lesson (lesson 6) teaches students how to solve problems with multi-digit answers that are not the same as the estimated quotient.

The '89 version and also the '91 version of the *stronger-pedagogy basal* program (see Lesson 4, Figure 1) introduced estimated quotients that are too large and estimated quotients that are too small. This instruction ocurred in only one lesson.

Problems with *multi-digit quotients* present new challenges to the student. The students must work the problem a part at a time. To do this, they must determine which part of the problem to work first. Sometimes the first part involves the first 2 numbers of the dividend, e.g., $24\overline{)972}$, and sometimes the first 3 numbers of the dividend, e.g., $24\overline{)1483}$. These problems with 2-digit quotients can be especially cumbersome when the estimated quotients are wrong and the students have to change the quotients.

Both the '89 edition and the '91 edition of the *stronger-pedagogy basal* introduced problems with 2-digit quotients and problems with 3-digit quotients in the same lesson. Some problems in that lesson had estimated quotients that proved to be incorrect.

The '88 version of the *higher-approval basal* devoted three lessons to the introduction of problems with multi-digit answers. The first lesson introduced problems with 2-digit quotients. The problems were limited to those in which the estimated quotient proved to be correct. The second lesson introduced problems in which an estimated quotient had to be changed, and the third lesson introduced problems with 3-digit quotients.

In the '91 version of the *higher-approval basal,* the lesson which introduces problems with 2-digit quotients remains similar to that of the '89 version. The next lesson, unlike the '89 version, does not teach students to work problems with estimated quotients that are incorrect. Instead, it introduces problems with 3-digit answers. Even though no mention is given in the teacher directions, 7 of the 19 problems in that lesson included estimated quotients that were incorrect.

One of the consequences of introducing concepts at too fast a rate is that two or more challenging concepts end up being introduced at the same time, causing confusion for some students. For example, in the higher-approval basal, students are initially taught to estimate by creating a simple division problem, taking the first digit of the divisor and the first two digits of the dividend; to work the problem $48\overline{)157}$, the students would create the problem $4\overline{)15}$. All the problems in the first lesson were designed so that this estimating algorithm would yield a correct answer. In the next lesson, the students use the same estimating procedure, but are introduced to problems in which the estimated quotient will be too big.

In the third lesson, problems in which the estimated quotient is too small are introduced in conjunction with a new rounding algorithm. Instead of looking at just the first digit of the divisor, the students are told to look at the first two digits and round to the nearest ten. The student text page and the accompanying teacher directions appear in Figure 2. As you read the excerpt from the student text and *Teacher's Guide,* note that the first time the students apply the new algorithm in problems, the estimate does not yield a correct quotient. The second time the new algorithm is introduced (in Example B), the new type problem (i.e., estimated quotient is too small) is presented, resulting in a potentially confusing situation for the students. The confusion results in part from the students having to learn to apply both the new rounding algorithm for estimating and the algorithm for correcting estimates that are too small at the same time.

The 1991 version kept the same sequence. In the first two lessons, the students rounded by keying just on the first digit of the divisor. The third lesson, the problematic one, was altered, though. Instead of beginning the lesson with an example in which $38\overline{)259}$ is rounded to $3\overline{)25}$, the lesson begins this way: $38\overline{)259}$ is rounded to $4\overline{)25}$. This change eliminates the confusion resulting from the introduction

Figure 1. Objectives For Each Lesson in the 2-Digit Divisor Chapter

The Stronger-Pedagogy Basal Objectives

Lesson	1989 Version	1991 Version
1	To use basic facts and mental math to find larger quotients.	To use mental math to find the quotients of multiples of powers of 10.
2	To use multiples of 10 as divisors to find 1-digit quotients.	To use estimation techniques of front end digits and compatible numbers to estimate quotients.
3	To find 1-digit quotients for 2-digit divisors. Estimating leads to a correct answer.	To find 1-digit quotients for 2-digit divisors. Estimating leads to a correct answer.
4	To change incorrect estimated quotients	To change incorrect estimated quotients.
5	To use mental math to solve word problems; to solve word problems having more than one step.	To divide by a 2-digit number when the quotient is a 2 or 3-digit quotient. Estimating sometimes leads to correct answer.
6	To divide by 2-digit divisors to find 2 and 3-digit quotients. Estimating sometimes leads to correct answer.	To choose an appropriate method of calculation (paper and pencil, calculator, mental math).
7	To divide by a 2-digit divisor to find quotients with zeros.	To divide whole numbers with zeros inthe quotient and money notation in the dividend.

The Higher-Approval Basal Objectives

Lesson	1989 Version	1991 Version
1	Divide by a two-digit divisor to get one-digit quotient. Estimating leads to correct answer.	Use mental math strategies to divide by a two-digit divisor.
2	Correct one-digit quotients that are to large.	Estimate quotients by using compatible numbers.
3	Round the divisor to make a better estimate for the quotient.	Divide by a two-digit divisor to get a one-digit quotient. Estimating leads to correct answer.
4	Divide by a two-digit divisor to get a two-digit quotient. Estimating leads to correct answer.	Correct one-digit quotients that are too large.
5	Correct two-digit quotients that are too small.	Round the divisor to make better estimates for quotients.
6	Solve problems by interpreting remainders.	Divide by a two-digit divisor to get a two- digit quotient. Estimating sometimes leads to correct answer.
7	Divide by a two-digit divisor to get a three-digit quotient.	Solve problems by interpreting remainders
8	Divide by a two-digit divisor to get a quotient with one or two zeros.	Divide by a two-digit divisor to get a three-digit quotient.
9	Solve problems by choosing addition, subtraction, multiplication, or division.	Divide by a two-digit divisor to get a quotient with one or two zeros.
10	Find missing factors.	Solve problems by choosing addition, subtraction, multiplication, or division.

of an algorithm that seems less efficient than the old strategy. However, students still must think about adopting a new rounding algorithm and at the same time learn to compare the remainder to the divisor, seeing if it is less than the divisor.

Coherence

Coherence refers to how the lessons in a chapter interrelate. What is taught in the first lesson of a chapter should prepare students for what comes later. Both the '91 versions of the basals place more emphasis on estimation to determine the reasonableness of answers, a worthwhile change from the '80s edition. Unfortunately, the manner in which the estimation is taught may make learning to work 2-digit divisor problems even more difficult. A lesson has been added at the beginning of the respective chapters that teaches students to use "compatible" numbers to estimate the answers to problems.

Compatible numbers are numbers that enable a student to divide and end up with either a single-digit or a multi-digit quotient in which all the numbers except the first digit are zeros. Here's an example of problems with compatible numbers: 7,200 ÷ 80, 350 ÷ 70, 9,000 ÷ 300.

In working with compatible numbers, the students are taught to round the dividend so that the quotient will end with zeros. To estimate $82\overline{)634}$, using compatible numbers, the students would change the problem to $80\overline{)640}$. To estimate $83\overline{)3{,}179}$, the students would change the problem to $80\overline{)3{,}200}$, and to estimate $28\overline{)8{,}905}$, the students would change the problem to $30\overline{)9000}$.

Introducing the estimating skill using compatible numbers at the beginning of the chapter may result in confusion for students when they apply the algorithm for calculating the exact quotient. In the calculation algorithm, the student always tries to find the multiple just smaller than the dividend. For example, to divide $31\overline{)144}$, the student would round off and figure $3\overline{)14}$ to get an estimated quotient of 4. However, using compatible numbers, the students would round to $30\overline{)150}$ to get an estimated quotient of 5.

The potential problem can be illustrated more specifically by examining the '91 version of the *stronger-pedagogy basal*. In the first lesson, students use mental math to find quotients of multiples of 10, such as 160 ÷ 80 and 2700 ÷ 30. The second lesson teaches students to use compatible numbers to find 1-digit quotients. The problems students are to work all have tens numbers as the divisor. The student text presents a model:

$70\overline{)375}$ — The compatible numbers 7 and 35 help you estimate the quotient.

All the problems the students are to work are written so that the estimated quotient can be found by making the first 2-digits of the dividend less to create a compatible number. In the example above, students change 375 into 35. To figure the estimated quotient for $40\overline{)218}$, the student writes $4\overline{)20}$.

Coherence problems begin in the next lesson, which introduces the calculation algorithm. The lesson begins with a review exercise in which the students are to estimate the quotient to a set of 15 problems. The problems appear in the "Quick Review" box that follows. Note that in 12 of the 15 problems, the students must use a compatible number that is larger than the dividend. For example, to estimate the

Figure 2. Making Better Estimates in Division

Student Material

A. Find 259 ÷ 38.

$$\begin{array}{r} 8 \\ 38\overline{)259} \end{array}$$
Divide.
THINK How many 3s in 25? 8

$$\begin{array}{r} 8 \\ 38\overline{)259} \\ 304 \end{array}$$
Multiply.
304 is greater than 259, so 8 is too big.

$$\begin{array}{r} 6\ \text{R31} \\ 38\overline{)259} \\ 228 \end{array}$$
You can make a better estimate.
THINK 38 rounds to 40.
How many 4s in 25? 6
Multiply.
Subtract and compare.
The remainder is 31.

B. Find 198 ÷ 28.

$$\begin{array}{r} 6 \\ 28\overline{)198} \end{array}$$
Divide.
THINK 28 rounds to 30.
How many 3s in 19? 6

$$\begin{array}{r} 6 \\ 28\overline{)198} \\ 168 \\ 30 \end{array}$$
Multiply.
Subtract and compare.
30 is greater than 28, so 6 is too small.

$$\begin{array}{r} 7\ \text{R2} \\ 28\overline{)198} \\ 196 \\ 2 \end{array}$$
Try 7.
Multiply.
Subtract and compare.
The remainder is 2.

Teacher Directions

Using the pages

Teach. Point out that in Example A, using the first digit of the divisor to estimate the quotient results in an estimate that is too large. Explain that rounding the divisor makes a more accurate estimate possible. In Example B, have students estimate without rounding first. Then work through this example using the rounded divisor. Point out that although this estimate is too small, it is closer to the quotient than the estimate which was made without rounding.

quotient for 107 ÷ 30, the students estimate 120 ÷ 30 = 4. In the lesson that follows this warm-up exercise, the calculation algorithm is introduced. Students are expected to first estimate the quotient using compatible numbers. Next, students are to use a *different estimating algorithm* (finding a multiple just smaller than the dividend). If the students use the compatible number strategy to get estimated quotients instead of finding a multiple just smaller than the dividend, they will obtain an incorrect estimated quotient for approximately one-third of the problems in the practice set. For example, for the problem 44 ⟌352, the compatible numbers would be 4⟌36, which would yield an estimated quotient of 9, which is not correct. Because the students have not yet been taught how to rework incorrect estimated quotients, the situation is likely to be frustrating for many students.

Quick Review

Estimate the quotient.

1. 395 ÷ 70	(6)	2. 714 ÷ 80	(9)	3. 456 ÷ 50	(9)
4. 157 ÷ 40	(4)	5. 237 ÷ 60	(4)	6. 533 ÷ 90	(6)
7. 817 ÷ 90	(9)	8. 639 ÷ 80	(8)	9. 118 ÷ 30	(4)
10. 466 ÷ 50	(9)	11. 623 ÷ 70	(9)	12. 218 ÷ 30	(7)
13. 107 ÷ 30	(4)	14. 531 ÷ 80	(7)	15. 345 ÷ 70	(5)

In the '91 version of the *higher-approval basal*, the entire second lesson is devoted to using compatible numbers to estimate quotients. Students work problems in which they make the dividend larger, 498 ÷ 49 is estimated as 500 ÷ 50, and problems in which they estimate by making the dividend smaller, e.g., 251 ÷ 39 is estimated as 240 ÷ 40. The next lesson, which introduces the calculation algorithm, does not mention compatible numbers. But the third lesson, which introduces problems where the estimated quotient is too large, shows the estimating step using compatible numbers. If, in fact, students used the compatible number algorithm, several problems would yield estimated quotients that are too large, e.g., 22⟌134 —> 20⟌140, which yields an estimated quotient of 7. (The correct quotient is 6.)

Clarity of Teacher Communication

The role of the teacher's manual is to help the teacher explain new concepts to the students in a clear, concise manner. When faced with a class of up to 30 students, the teacher might like to refer to suggestions about what to say and do to ensure that she is communicating in a clear manner that facilitates student understanding. Neither program provides specific suggestions.

Explanations. The 1989 version of the *stronger-pedagogy basal* relied mainly on the teacher to explain problems that were illustrated in the student text. A typical direction to the teacher appears below. It's taken from the page on which the calculation algorithm is first introduced.

> *Lesson Development* *Discuss each instruction box in the steps for finding the answer. In the second step, emphasize that rounding the divisor makes the problem like the ones in the previous lesson. Caution students to be careful to multiply the quotient by the actual divisor, not the rounded divisor. Discuss the steps used to check the answer. Also point out that the answer 5 seems reasonable since 40 x 5 = 200.*

The 1991 the *stronger-pedagogy basal* took much of the responsibility from the teacher for explaining concepts clearly, in what appears to be the hope that students will be able to explain the concepts to their peers more clearly than the teacher can.

At the beginning of each unit in the '91 edition is a section entitled *Communication*. In many of these exercises, the teacher has the students hypothesize how to solve a problem before the teacher actually explains the strategy. Below is an exercise that appeared before the introduction of the lesson on correcting estimates.

> *COMMUNICATION*
> **Writing in Math** Write the division problems 56⟌340 and 93⟌450 on the chalkboard. For each problem, have students estimate its quotient, multiply to verify the accuracy of the estimate, and write a paragraph explaining their findings and how they would handle the situation. [Possible answer: The estimated quotient of the first problem will be too small and the estimated quotient of the second problem will be too large.]

Activities such as these may function well for higher performing students, but for lower performers they may not function as intended. These students need carefully controlled explanations and active involvement.

Manipulative activities. A fundamental objective of any math program is to instill in students a clear understanding of the events signified by a mathematical operation. Manipulative activities should provide a framework for understanding, but not interfere with the teaching of the algorithm itself. The area of manipulative activities represents the greatest differences between programs and between editions. The *higher-approval basal* drastically reduced its manipulative activities while the *stronger-pedagogy basal* dramatically increased its manipulative activities.

In the two division chapters (1-digit divisor and 2-digit divisor problems), the *stronger-pedagogy basal* (1989 version) had only one lesson in which the students used manipulative material. In that lesson, the students worked with play money and had to divide a given amount among a specified number of students. The exercise focused on place value. The problem required the students to trade in a hundred for 10 tens or a ten for 10 ones to divide the money. A model of how to divide money was given: "For 3 into 432, we give a hundred to each, we have a hundred left over; so we change the hundred into 10 tens, then divide the tens," etc. The exercises the students were to do independently were more difficult than the model, but probably in a reasonable range; e.g., the model showed 3 into 432, but students were required to work independently 5 into 855 and 4 into 276, both of which require the students to change more than one 100 into tens.

The 1991 the *stronger-pedagogy basal* introduced a new section in each lesson. The section entitled Explore and Connect included a variety of activities, some using calculators and other materials. Here are the activities included in the lessons in the single-digit divisor chapter.

- Students work in groups using a map to plan a 200-mile bike trip. They plan their route, then determine the number of days the trip would take if they traveled the same realistic number of miles each day.
- Students work in pairs with a calculator using repeated subtraction to work problems.
- Students work in groups using play money to model division problems.
- Students work in groups using guess-and-check strategy to determine averages.

The potential problem with these activities is that they are very time consuming.

There was also a significant difference between the '89 versions of the *stronger-pedagogy basal* and the '88 version of the *higher-approval basal* in the amount of manipulative activities for division. In the '88 version of the *higher-approval basal*, three lessons (two for 1-digit divisor problems and one for 2-digit divisor problems), had the students work with place value units (hundreds, tens, and ones), to solve division problems. The students work a total of 47 problems:

- 28 problems with a single-digit divisor and a 1-digit quotient, 9 | 80.
- 12 problems with single-digit divisors and 2-digit quotients, e.g., 3 | 157.
- 7 problems with 2-digit divisors and single-digit quotients, e.g., 21 | 171.

The students were to work problems with manipulatives and then to record their work using the form of the algorithm. After working a set of problems this way, students were to work problems just using the algorithm. The *Teacher's Guide* was quite unclear as to the manner in which the teacher would lead the students through the transfer from the use of manipulatives to the use of the algorithm. The following directions to the teacher appear for explaining the model of the algorithm illustrated in the student text:

> *Practice.* For the example after Exercise 6, students should recognize that 3 is written in the ones place of the quotient because there are 3 groups of 48 in 157, rather than 3 groups of 4 in 15, which would lead them to write 3 in the tens place.

In the '91 version of the *higher-approval basal*, the number of problems students are to work with manipulatives is drastically reduced, only two problems with single-digit divisors and two problems with 2-digit divisors are presented. More structure is provided in the form of a worksheet that asks questions such as:

> To divide 83 by 3, can 8 tens be divided into 3 groups?
> How many tens will be in each group? 2
> After dividing, trade 2 tens for 20 ones.
> Can 23 be divided into 3 groups?
> How many ones in each group? 7
> Is there a remainder? Yes
> After dividing, 2 ones remain.

Although manipulatives are indispensable in developing basic number concepts, the need for using manipulatives to work complex algorithms is far less clear. This confusion is reflected in the positions taken by the two basal programs in their '80s edition and how each publisher revised its position in its 1991 edition.

Guided Practice

Many students need a transition between the explanation given in the introduction and the problems to be worked independently. In guided practice, which occurs after a concept is introduced, the teacher asks questions that prompt appropriate student application of the new concept (Good, Grouws, & Ebmeier, 1983). As the students approach mastery, teachers should decrease the level of prompting until the students are functioning independently (Paine, Carnine, White, & Walters, 1982).

As a general rule, structure should be provided to facilitate a student success rate of at least 70-80 percent. A success rate can be calculated with this fraction:

$$\frac{\text{Number of problems worked correctly}}{\text{Number of problems attempted}}$$

As noted earlier, basals offer rather vague explanations for introducing new concepts. After these initial explanations and activities, students are expected to work several problems on their own without explicit guidance from the teacher. Neither the

Fifth Grade Division—*Continued*

higher-approval basal nor the *stronger-pedagogy basal* provide suggestions for conducting guided practice. No specific wording suggested that the students utilize the steps that are modeled in the student text. In both versions of the *stronger-pedagogy basal* and the *higher-approval basal*, the major part of the guided practice section of the *Teacher's Guide* alerted the teacher to common errors the students might make.

Practice

Initial practice. Practice that appears immediately after the introduction of a concept should be coordinated with the types of problems presented in teacher-directed activities. As a general rule, the teaching in structured exercises should prepare the students for the problems they'll encounter in independent work. Textbook authors can put in problems that test students' generalization, but the generalization should be reasonable.

Sometimes programs expect too much from the students. For example, with 2-digit divisor problems, some problems will yield estimated quotients that may be two or three numbers from the correct quotient (i.e., $13\overline{)93}$). The '89 edition of the *higher-approval basal* had a lesson in which this type appeared without any teacher guidance. Remember that the *higher-approval basal* initially teaches students to estimate by looking only at the first digit of the divisor, e.g., for $30\overline{)244}$, the estimation problem is $3\overline{)24}$. Using this estimation algorithm, the students will encounter several problems, $38\overline{)278}$, $13\overline{)93}$, $39\overline{)248}$, and $29\overline{)203}$, that yield estimated quotients that are 2 or 3 numbers too great. The '91 version is the same, except the *Teacher's Guide* does contain this direction to the teacher: "Stress that students must sometimes try several digits in the quotient before they find one that works" (p. 156).

Later practice. Distributed practice enables a student to become fluent in working problems. After students are able to work a new type problem with relative ease, they might benefit from discriminated practice in which problems of the recently introduced type are integrated with problems of previously taught types. For example, after students learn to work 2-digit divisor problems, they might become confused about where to write the first digit of the quotient in 1-digit divisor problems. Neither program provides any *mixed practice sets* that include 2-digit and 1-digit divisor problems. The only activities that provided practice in this discrimination were story-problems which included only one of each type problem.

Review

Review is the means by which students receive the practice needed to retain what they have learned. Review can take the form of computational problems, or the problems can be integrated as a component within a more complex context—for example, a story problem that can be solved by dividing with a 2-digit divisor.

The review sequences in both programs would be inadequate for many average and lower-performing students. In the '91 version of the *higher-approval basal*, a chapter on geometry follows the chapter on 2-digit divisor problems. The *Teacher's Guide* for the geometry chapter makes no reference to 2-digit divisor problems. The chapter following geometry presents addition and subtraction of decimals. This chapter also contains no instructions to the teacher for reviewing 2-digit divisor problems. The first reference to 2-divisor problems appears in a cumulative review test which appears at the end of the addition and subtraction of decimal chapters—a point which would probably occur several weeks after the students had finished the 2-digit divisor chapter. One problem that tests the students' knowledge of the 2-digit divisor algorithm appears in the test. The problem tested,

$$\begin{array}{r} 200\text{ R1} \\ 16\overline{)3201} \end{array}$$

is a difficult type, and the remediation spelled out in the *Guide* directs the teacher only to the page where difficult-type problems like this with zeros in the quotient are taught. The first massed review of 2-digit divisor problems occurs in the next chapter on a page that appears 78 pages after the end of the divisor chapter.

In the '91 edition of the *stronger-pedagogy basal*, a chapter on fractions follows the 2-digit divisor chapter. Only two problems (both story problems in the middle of the chapter) review 2-digit divisor problems. A cumulative review test at the end of the chapter does include items testing the 2-digit divisor algorithm.

Conclusion

In an ideal world, each revision of a program would improve its suggestions where students encountered difficulty in the prior edition. Each revision would result in a more effective instructional tool. Unfortunately, the 1991 versions of these programs did not make changes that would significantly ameliorate the major short comings we found in the 1980's editions. Three lessons will not give many students enough time to learn the calculation algorithm. Moreover, the rapid introduction of concepts results in a new rounding algorithm and a new problem type (i.e., estimated quotient is too small) being introduced at the same time. In fact, the '91 version of the *higher-approval basal* is in such a rush that 19 problems with incorrect estimated quotients were introduced before students were taught how to work that type of problem.

Besides the difficulties caused by the rapid rate of introduction, many students will be confused by contradictory algorithms. For example, with the compatible-number algorithm, students often round up whereas with the calculation algorithm, they round down. The new editions of the *stronger-pedagogy basal* avoids confusing teacher explanations by having the students do the explaining. Finally, the programs lack guided practice and adequate review.

The 1991 the *higher-approval basal* did make some improvements compared to its 1989 edition. Activities from the '89 edition that seemed to be potentially confusing or too time consuming have been eliminated or revised. On the other hand, the 1991 *stronger-pedagogy basal* reflects changes from the 1989 program which could make it a less effective tool, especially for the less-experienced teacher.

As noted in the introduction, our interest was in fundamental aspects of pedagogy, not the new NCTM *Standards* in and of themselves. The '91 edition of the *stronger-pedagogy basal* added many activities to reflect the new NCTM *Standards*. The *higher-approval basal* also has many activities that are consistent with the *Standards*. The central point, however, is that if students are not able to divide, the failure they experience will not lead them to value mathematics, reason mathematically, communicate mathematics, or solve problems. ◈

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Manipulatives—The Effective Way

by David Evans
Western Australia College of Advanced Education
Douglas Carnine
University of Oregon

In the past ten years, some meta-analyses (e.g., Sowell, 1989) have provided minimal support for using manipulative materials in elementary school mathematics. For the most part, the studies included in those reviews compared instruction with manipulatives and instruction without manipulatives.

Research by Resnick and Omanson (1987) suggests that mathematical proficiency with concrete materials does not result in the same proficiency with symbolic representations. In fact, the results reported by Resnick and Omanson would indicate an inverse relationship: greater proficiency in using concrete representation was paired with weaker proficiency in using a symbolic representation, and vice versa.

It would appear that instruction may be required to ensure that both procedural and conceptual knowledge are developed. What is not clear is which type of instruction and which representations best promote this relationship (Baroody, 1989).

Resnick and Omanson (1987) allude to a possible solution to this dilemma in their concluding comments. They question whether blocks or manipulatives play a crucial role in learning borrowing in subtraction and suggest that "perhaps any discussion of quantities manipulated in written arithmetic, without any reference to the block analogue, could be just as successful in teaching the principles that underlie written instruction" (p.90). Alternatively other representations could be used (e.g., pictorial representations and symbolic representations). Lesh et al. (1987) take this idea one step further and suggest that perhaps a combination of representations could be used to develop concepts and procedures in mathematics.

The primary focus of the present study was on the efficient and effective development of procedural and conceptual mathematics knowledge. Two representational modes—concrete and symbolic—were used to teach two-digit minus two-digit subtraction with borrowing.

The strategies were drawn from two commercial texts. The first strategy, labelled the manipulative strategy, was adapted from the mathematics series *Explorations* (Addison-Wesley, 1988) which also includes the very popular *Math Their Way*. This strategy advocates the use of a variety of concrete objects with which students first explore the concept of borrowing and then are guided by the teacher through the steps for solving subtraction problems. Students are explicitly shown how problems are worked using the manipulatives and how this transfers to a symbolic form. For example, students "see" how a ten is borrowed and decomposed into ten ones. This visual image is later linked to a symbolic representation of the problem as students work the problem with the concrete material and record their work in a symbolic mode.

The second strategy was developed from the mathematics series *Connecting Mathematics Concepts* (Engelmann & Carnine, 1991). This strategy, labelled the algorithm strategy, required students to complete all work in a written symbolic representation. Prior to learning how to work problems, students were taught the conceptual elements that underlie the rules used in borrowing. For example, the teacher modelled, then students practiced how to rewrite numbers in differing expanded notation form (e.g., 48 equals 40 plus 8, and 30 plus 18). Students then used this rewriting rule to work subtraction problems requiring borrowing.

Each student received instruction using both strategies. The order in which strategies were taught was counter balanced: (1) either the manipulative strategy followed by the algorithm strategy; or (2) the algorithm strategy followed by the manipulative strategy. Prior to strategy teaching, students were taught how to compute minus-9 facts in a manner consistent with the first strategy they would learn, i.e., the manipulative group learned to compute minus-9 facts using manipulatives, while the algorithm group learned an algorithm for computing minus-9 facts.

Throughout the study, data were collected to address the following issues:

- The efficiency of instruction as measured by the amount of instructional time.
- The effectiveness of instruction as measured by posttest and maintenance scores.
- The conceptual understanding students attained, as indicated by interviews with students before, during, and after the study.

Method

Subjects and Setting

Twenty-six students in two second and third composite grades from a school in the Pacific Northwest participated in the study. The school, designated a Chapter 1 school, was situated in a low socioeconomic neighborhood.

Students in each of two classrooms were first divided into two groups: (1) Chapter 1 students and (2) special education students. Each group was in turn divided into female and male subgroups. Using the mathematics subtest of the *California Test of Basic Skills*, percentile differences between groups were analyzed using a one-way analysis of variance. No significant differences were found ($p > .44$). Groups were then randomly assigned to an instructional group: Manipulatives first (MN) or algorithm first (AL).

Table 1. Breakdown of Students Included in the Final Analysis by Gender and Type of Educational Services Received

Subject	Group	
	Manipulatives First (MN)	Algorithm (AL)
Male	5	8
Female	8	5
Total	13	13
Regular Education*	11	10
Special Education	1	2
Speech & Hearing	1	1

* All students received Chapter 1 services.

Measures

Subtraction problems screening test. The subtraction problems screening test consisted of eight subtraction problems; six problems required borrowing, two did not. Students who correctly answered five or more subtraction problems were not included in the final data analysis. This decision was corroborated through student interviews prior to treatment.

Place-value preskill assessment and instruction. The place-value preskill assessment and instruction focused on student understanding of place-value involving tens and ones. Students were presented randomly selected numbers represented with concrete objects (Dienes blocks that use color and size to represent place values). They were asked to count and state the quantity represented by the concrete objects. Students were also asked questions designed to assess their understanding of place-value of written numerals. Students who made errors received instruction until they correctly answered similar items.

Subtraction probes. Subtraction probes were given each day during the study. Each probe consisted of six subtraction problems, five requiring borrowing and one not.

Student interviews. A qualitative measure, adapted from the informal study conducted by Resnick and Omanson (1987), was used to capture student understanding of concrete and symbolic representations. Its focus was the students' conceptual understanding of borrowing. This qualitative measure was conducted three times: (1) prior to the study as part of the screening process, (2) at the conclusion of Phase 1, and (3) at the conclusion of Phase 2.

Students worked one problem with a symbolic representation and one with a manipulative representation. What the student said was recorded on audio-tape, and physical actions were noted on a checklist by the tester. At specific points when working the problems, the observer asked the student a question that required demonstration of conceptual knowledge to justify a particular procedural action.

Consumer satisfaction. At the completion of each phase, students were interviewed as to how they felt when using the representation in the immediately preceding instruction. Students were asked what they thought was easy and what was difficult about the representation. At the end of the study, students were also asked which representation they liked better, why they liked it, and what they did not like about the other representation.

Lesson duration. All lessons were timed by the teacher. The teacher commenced timing with a stopwatch when the lesson began. Timing continued until the teacher completed that day's instruction.

Twenty-four lessons out of the ninety lessons constructed for this study were observed. The data on duration of lessons was compared with the times noted by the teacher conducting the lesson. Agreements, defined as the two times within 15 seconds of each other, were found on 21 of the 24 lessons observed.

The three disagreements were made in the first five lessons of Phase 1. Checks on lesson duration timings were maintained throughout the study to ensure that discrepancies did not recur. At the conclusion of the study, the average discrepancy in lesson duration timings was 8.25 seconds per lesson.

Posttest. The day after the completion of the intervention, each student was individually administered a Posttest that involved completing five subtraction problems using a concrete representation (four problems required borrowing) and five problems using a symbolic representation (four problems required borrowing). The order in which the representations were presented was counter-balanced within each group.

Maintenance Test. A Maintenance Test was given three weeks after the conclusion of the intervention. The Maintenance Test was constructed and administered in the same manner as the Posttest, except the Maintenance Test was counterbalanced in relation to the order from the Posttest.

Procedures

The study was conducted within the regular sequence of the mathematics curriculum. Instruction was provided by graduate students with teaching experience.

Symbolic representation in Phase 1. Students were taught in small groups of three to five. Before learning to borrow, students were pretaught three component skills of the strategy: place-value addition, rewriting for borrowing, and deciding when to borrow. The teaching began with the teacher directly modeling a component skill. Teacher involvement was gradually faded until students were able to perform the three component skills independently. Borrowing was then introduced. (Details of the intervention can be found in Evans, 1990.)

The mastery criterion for students using the the symbolic strategy was correctly answering four out of five subtraction problems involving borrowing on lesson probes for two consecutive days. On at least one of these two days, students had to answer the nonborrowing probe correctly. (Students were given a 10-minute time limit in which to complete the six problems to accommodate school administrative concerns.)

Concrete representation in Phase 1. The concrete representation strategy was drawn from *Explorations* (Addison-Wesley, 1987). The authors of the program state:

". . . work with concrete materials is critical to concept development" (p. 285). The instructional strategy used comprised four steps. (Details of the intervention can be found in Evans, 1990).

Step 1. *When to share*
Step 2. *Breaking up and taking away*
Step 3. *Working together (or guided practice)*
Step 4. *Finishing the story (or independent work)*

As students completed the lesson probes, the instructor monitored student work to ensure that the concrete materials were being used correctly. Students did not attain mastery unless they were able to display evidence that the manipulatives were being used correctly. The mastery criterion for students using the concrete representation was the same as the set for the symbolic representation.

Contrasting representation in Phase 2. After reaching criterion with the first representation in Phase 1, the students in a group were taught to solve borrowing problems using the other representation (Phase 2). The MN group was taught to use the symbolic representation but was allowed to use ones counters to answer fact problems. The AL group was taught to use a concrete representation exactly as was given to the MN group in Phase 1.

After meeting criteria in Phase 2, students completed the posttest and the consumer satisfaction interview. Three weeks later, students completed the maintenance tests. The interviews to assess student understanding of the two representations were conducted prior to the intervention and after each phase.

Daily decisions and record keeping. Students who attained criterion in Phase 1 did not participate in instruction for the next lesson, but engaged in other independent work. Independent work consisted of material that was not related to subtraction, but had been introduced in previous math lessons. Lesson duration was not recorded for these students. However, instructional probes continued to be given to the students on each day of instruction. Failure to maintain criterion on two consecutive days required students to return to the intervention and attain criterion again.

Fidelity of instruction. Instruction was conducted by three graduate students who worked from scripted lesson plans. Instructors were coached in the appropriate use of the lesson plans, which were practiced until instructors presented them accurately and efficiently. During the study, the three instructors were rotated daily to prevent any significant effects due to the instructor. The observer who checked lesson duration also monitored fidelity of implementation. A checklist of teacher behaviors was used to check teacher adherence to the lesson plans, pacing of the lesson, and conformity to the time schedule. This check was conducted daily until no violations were found in implementation.

Results

Maintenance and Posttest Results

Procedural proficiency with the concrete representation. Posttest and Maintenance Test data for each group for solving subtraction problems requiring borrowing with a concrete representation are shown in Table 2. The differences between the MN and AL groups were not significant.

Table 2. Mean and Standard Deviation for Posttest and Maintenance Test Using Concrete Representations

Group	Posttest			Maintenance Test		
	N	M	SD	N	M	SD
Manipluatives First (MN)	13	3.23	2.16	12*	2.75	2.22
AlgorithmFirst (AL)	13	3.54	1.85	12*	3.25	1.48

*One student left school between the posttest and maintenance test

Table 3. Mean and Standard Deviation for Posttest and Maintenance Test Using Symbolic Representation

Group	Posttest			Maintenance Test		
	N	M	SD	N	M	SD
Manipluatives First (MN)	13	2.77	2.24	12*	1.42	1.73
AlgorithmFirst (AL)	13	2.69	2.36	12*	2.75	2.26

*One student left school between the posttest and maintenance test

Procedural proficiency with symbolic representation. Procedural proficiency in using a symbolic representation was evaluated by having students complete problems using paper and pencil and without the use of aids, e.g., number line or counters. The mean and standard deviation for each test by group is shown in Table 3. The differences were not significant.

Lesson Duration

A summary of data for length of instruction for Phase 1 and Phase 2, and the combined time for both of these phases, is shown in Table 4. The mean duration of instruction for each group is also shown in Figure 1.

A two-way repeated-measure ANOVA was conducted, with Phase of Instruction and Treatment group as the independent variables, and the duration of instruction as the dependent variable. The result of this analysis indicates that there is a statistically significant interaction between Group and Phase, $F(1, 48) = 6.54, p < .03$.

An analysis was conducted of the simple main effects to determine what contributed to this interaction. The MN and AL groups differed significantly for Phase 1, $F(1, 48) = 9.67, p < .05$, but this significant difference was not found in Phase 2. This result suggests that learning to borrow first with a symbolic representation leads to more efficient learning than beginning with a concrete representation.

A second significant simple main effect occurred in relation to the order in which groups learned to borrow using concrete and symbolic representations, $F(1, 48) = 6.56, p < .05$. These results showed that the time required to learn to borrow using a concrete representation first (Phase 1) was significantly greater than the time for learning the symbolic representation next (Phase 2). In contrast, when learning to borrow using a symbolic representation was followed by using a concrete representation, there was no statistically significant difference in the duration of instruction for the two phases.

Understanding of Symbolic Representation

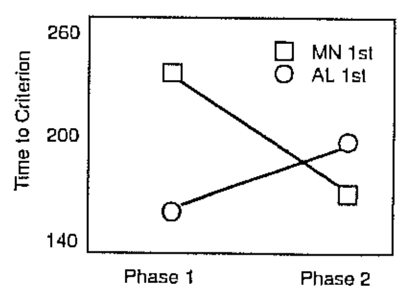
Categorical data included actions that could be rated as occurring or not occurring, such as students crossed out the tens digit in the minuend and wrote the number that was one less. (The categories are shown in the left column of Table 5.)

Categorical data were rated in three ways. Students who failed to complete a particular category or who showed no evidence of completing a category were rated as "Don't Know" or "DK." A response was recorded "yes" for each student who completed a category as described, while a response that did not

Table 4. Mean Standard Deviation, and Range for Duration of Lessons in Phase 1, Phase 2, and Total for Phase 1 and Phase 2

Group		Phase 1	Phase 2	Phase 1 + Phase 2
MN:	M	242.81	169.30	412.10
	SD	50.21	50.23	83.46
Range		(171.55-315.98)	(67.54-234.80	(255.71-508.70)
AL:	M	153.61	183.85	337.46
	SD	52.88	112.70	162.81
Range		(86.18-267.33)	(29.03-340.68)	(115.22-608.02)

Figure 1. Mean Duration of Instruction to Criterion for Phases 1 and 2



match the description was marked as "No." The percents for each category after Phase 1 and after Phase 2 appear in Table 5.

After Phase 1. In the Interview, conducted at the conclusion of Phase 1, 69.2% (9 out of 13) of the AL students correctly answered the specified problem. This percentage of correct answers was a noticeable improvement from the screening interview, where only one student from the AL group was able to respond correctly. The improvement was not as clear for students in the MN group who had just completed instruction using a concrete representation. Correct answers were attained by 46.2% (6 out of 13) of students in the MN group. The student from the MN group who had answered the problem correctly in the screening interview was unable to answer the problem after Phase 1. She again tried to count from the subtrahend to the minuend but became confused. In an attempt to obtain the correct answer she asked on more than one occasion to use the blocks. Her final response was that it could not be done without blocks.

Most students no longer inverted the numerals in the ones column when working the problem after Phase 1. It would appear that in learning to borrow, through a symbolic or concrete representation, the students learned to take the subtrahend from the minuend.

Although differences between groups in Interview 1 were not great, the AL group tended to score higher than the MN group in all categories. It appeared that a majority of MN students were unable to transfer what they had learned with the concrete representations to the symbolic form.

Qualitative comments pertaining to the process of borrowing were more straightforward with the AL group, which had just learned the symbolic strategy. The question "why did you take one from [the tens] column and put it [in the ones column]" brought an almost unanimous response from the students that "the big number was not on the top."

MN students had greater difficulty answering the same question. Only 30.7% of MN students borrowed, but none was able to explain why. All but one of the 30.7% of MN students were able to complete the problem correctly. The one student who responded incorrectly made a mistake with the fact calculation in the tens column. Another 23% of MN students stated that they could not take nine away from two.

Three of the seven MN students who calculated an incorrect answer were asked to work the same problem using the Dienes blocks. Two students calculated the correct answer using the Dienes blocks, while the third made a counting error with the ones blocks, and was off by "one" in his answer. After working the problem with the concrete representation, students were asked to compare their answer to the one they had calculated using the symbolic representation. Each student recognized they had calculated answers that were not the same. When asked which answer was correct, all three emphatically indicated the answer worked with blocks. The reason given was that "you can't work these problems without the blocks." One student went further and stated, "You can't change the numbers (borrow) without the blocks . . . ." These responses suggested that students were unable to make the link between the concrete and symbolic representations. Specifically, one student verbalized his inability to "see" borrowing a ten with blocks when using a symbolic representation.

Table 5. Categorical Data from Interviews on Symbolic Representation After Phase 1 and After Phase 2.

Category		MN Phase 1	MN Phase 2	AL Phase 1	AL Phase 2
Starts problem in ones column	Yes	76.9%	100%	100%	100%
	No	15.4%	0%	0%	0%
	DK	7.7%	0%	0%	0%
Borrows a ten	Yes	61.5%	69.2%	84.6%	76.9%
	No	23.1%	23.1%	15.4%	7.7%
	DK	15.4%	7.7%	0%	15.4%
Indicates 10 has been borrowed	Yes	61.5%	53.8%	84.6%	76.9%
	No	23.1%	23.1%	15.4%	23.1%
	DK	15.4%	23.1%	0%	0%
Indicates borrowed 10 is placed in ones	Yes	69.2%	76.9%	84.6%	100%
	No	15.4%	23.1%	15.4%	0%
	DK	15.4%	0%	0%	0%
Ones: minuend minus subtrahend	Yes	69.2%	76.89%	84.6%	100%
	No	15.4%	23.1%	15.4%	0%
	DK	15.4%	0%	0%	0%
Tens: minuend minus subtrahend	Yes	84.6%	100%	100%	100%
	No	0%	0%	0%	0%
	DK	15.4%	0%	0%	0%
Correct Answer	Yes	46.2%	38.5%	69.2%	69.2%
	No	53.8%	61.5%	30.8%	30.8%

DK = Don't Know

After Phase 2. The categorical data after Phase 2, shown in Table 5, reveal several trends. The AL group appeared to perform slightly better in all categories than the MN group. Except for three categories ("borrows a ten," "indicates that 10 has been borrowed," and "correct answer"), *all* AL students were observed to have performed the described behavior. First, 76.9% of students in the AL group showed that they took a ten away from the tens column. The same ten students then indicated that the ones column increased by ten ones. All but one of these ten students who had borrowed a ten and increased the number of ones by ten completed the problem correctly; the one incorrect computation was the result of a fact error in the ones column.

The MN group worked the problem after Phase 2 in a similar fashion, but at a level slightly below that of the AL group. Of the 69.7% of MN students who borrowed a ten, all but one showed this by rewriting the ones numeral. However, only a third completed the problem correctly. Fifty percent of MN students who borrowed and rewrote the ones numeral made a fact error in the ones column (i.e., all wrote that 16 minus 9 equals 6).

Students' understanding of why they had borrowed varied from group to group. Forty-six percent of the students in the AL group stated that they had borrowed because they could not take the subtrahend from the minuend. Only 23% of MN students verbalized this same response.

While students in both groups showed evidence of borrowing (e.g., writing a one next to the numeral in the ones column, or rewriting the minuend with a number 10 or larger), students were unable to always state the place-value of the one they borrowed. In the AL group 39% of the students said it was a "ten." The response given by 31% (4 of 13) of the MN students was "16." Two other MN students were able to isolate the place value of the one and say it was a "ten."

The algorithm "bug" of inverting the ones digits instead of borrowing was not apparent after Phase 2. No AL students exhibited this behavior. The inversion of the subtrahend and minuend in the ones column was observed for 30.1% of MN students. This result may indicate that students who first received instruction using a concrete representation followed by symbolic representation still retained this bug, whereas students who received instruction using a symbolic representation then a more concrete representation were less likely to retain this bug.

Summary of symbolic understanding. In comparison to the screening interview, both groups of students demonstrated a substantial increase in ability to borrow after training. The MN group did not exhibit the behavior of borrowing a ten symbolically as frequently as the AL group, but the number of MN students showing this behavior was still considerable. The MN students were not able, however, to explain their work as clearly as the AL students and made numerous fact errors in completing the problem. It was apparent that MN students relied heavily on the use of concrete objects to work basic facts.

The results after Phase 2 showed that 62% of the AL students were able to correctly complete the symbolic problem while only about half as many MN students correctly completed the problem. The amount of understanding that was verbalized varied, with the AL group showing more evidence of conceptual understanding and proficiency in solving problems symbolically.

Understanding of Concrete Representation

After Phase 1. The results are summarized in Table 6. All MN students read the problem and represented the minuend with Dienes blocks correctly. All students traded a ten for ten ones, and were able to verbalize that they made this trade because they did not have enough ones to take away 9. A correct answer was computed by 92.3% of MN students. The one incorrect response was by a student who borrowed a ten for ten ones, but was unable to take away the correct amount (i.e., she was unable to take away any blocks as she appeared not to know what to do).

In contrast to the MN group after Phase 1, students in the AL group were unable to complete many categories for borrowing with concrete objects. Although 92% of the AL students were able to represent the minuend with concrete objects and did not invert the minuend and subtrahend in the ones column, only 15% showed that they borrowed a ten to obtain ten ones. Examination of where students began working the problems showed that only 15% of the AL students began in the ones column.

Students who began in the tens column took the correct number of tens away, but when faced with taking nine ones away, they invented their own solution for coping with an insufficient number of ones. These students took another ten away and placed a one from the reserve pile in the ones column. This gave the students the correct answer, but students were not aware that they had not conserved the original amount in the minuend.

A few AL students who responded incorrectly when using the concrete representation were asked to work the same problem using a symbolic representation. Students who calculated a correct answer were asked to compare their answer to the answer obtained with manipulatives and explain the different answers. Some students stated that "you can't borrow using blocks," while others stated that they orrowed while using the symbolic representation ut did not when using the concrete representation. . few students who were asked whether you could orrow using blocks said "yes," but did not know ow this could be achieved using the blocks. These esponses appear to indicate that students who earned to use a symbolic representation found it ifficult to link this knowledge to the use of Dienes locks.

After Phase 2. After Phase 2 most students were ble to borrow correctly. The majority of students raded a ten for ten ones (MN group 92.3% and AL roup 100%). One MN student placed ten ones in the nes column, but failed to conserve the original uantity by removing a ten.

No evidence of inverting the minuend and sub-rahend was evident by any students after Phase 2. Correct answers were calculated by 61.5% of the MN tudents and 84.6% of the AL students.

Summary of concrete understanding. Phase 1 produced a dramatic increase in the number of stu-lents in the MN group who could complete the problem correctly, as well as a fair understanding of he concepts involved. The AL group, however, did not show the same breadth of procedural knowledge. They were aware that they had to borrow but could not connect or link the conceptual understanding learned using a symbolic representation with a concrete representation.

For the MN group, procedural knowledge after Phase 2 was comparable to after Phase 1; however only 62% of students were able to calculate the correct answer, as compared to 92% after Phase 1. In contrast, the AL group showed a large increase in conceptual and procedural knowledge (23% to 85%) from Phase 1 to Phase 2. It seems that learning to borrow using a symbolic representation, then a concrete representation, lead to better understanding of the role of place value in borrowing.

Table 6. Categorical Data from Interviews on Concrete Representation after Phase 1 and Phase 2.

Category		MN Phase 1	MN Phase 2	AL Phase 1	AL Phase 2
Reads the problem	Yes	100%	100%	100%	100%
	No	0%	0%	0%	0%
Shows concrete representation	Yes	100%	100%	92.3%	100%
	No	0%	0%	7.7%	0%
Starts in the ones column	Yes	92.3%	92.3%	15.4%	84.6%
	No	7.7%	7.7%	84.6%	14.4%
	DK	0%	0%	0%	0%
Trades a ten for ten ones	Yes	100%	84.6%	15.4%	84.6%
	No	0%	15.4%	61.5%	14.4%
	DK	0%	0%	23.1%	0%
Ones: minuend minus subtrahend	Yes	92.3%	84.6%	15.4%	100%
	No	0%	15.4%	61.5%	0%
	DK	7.7%	0%	23.1%	0%
Tens: minuend minus subtrahend	Yes	92.3%	92.3%	61.5%	100%
	No	0%	7.7%	7.7%	0%
	DK	7.7%	0%	30.8%	0%
Correct answer	Yes	92.3%	61.5%	23.1%	84.6%
	No	7.7%	38.5%	76.9%	14.4%
	DK	0%	0%	0%	0%

DK = Don't Know

Consumer Satisfaction Interviews

After Phase 1

Students in both groups appeared to enjoy using the representation they had been introduced to in Phase 1. Common responses included: "fun," "good," and that they "liked it." For example, students in the AL group enjoyed rewriting numbers and crossing them out.

Few specific comments were made about what was hard, or was not liked. The majority of students in both groups said that nothing was hard. The few negative comments included "scared because they didn't know how to do it," "confused," "nervous," and "strange." Several students (12.5%) expressed that when they first started to learn to borrow it was hard, but that they soon found it easier when they were successful at solving problems.

After Phase 2

Students did not appear to express any dislike of the second representation. Comments of "fine," "liked it," "good," and "excited" were given in response to how they felt. Negative comments were also recorded. One AL student stated that he "hated" working with concrete objects as it was slower than working the symbolic representation. One MN student said it was too hard using the symbolic representation, and she would like to use blocks all the time. When students were asked to compare the two representations they used, students in each group gave what appeared to be contrasting responses. The student preferences are summarized in Table 7. Eighty-five percent of the students in the AL group strongly favored using symbolic representations. Reasons given were that using a symbolic representation was faster and more fun. The one student who responded in favor of using a concrete representation said it was "too difficult to remember borrowing in your head."

Table 7. Preferences for Representations

Response	MN	AL
Concrete	61.5%	7.7%
Symbolic	30.8%	84.6%
Both	7.7%	0.0%
No Response	0.0%	7.7%

Sixty-two percent of MN students who learned to use a concrete representation first and then a symbolic representation were in favor of using concrete objects. They favored the use of blocks because it was easier when "numbers were too big" and when "you don't have enough fingers." Half of the 31% of MN students who said they wanted to use symbolic representations indicated that they did not want to use blocks to work problems. The other MN students indicated that they did not want to use tens and ones blocks, but preferred fingers or counters.

No matter what representation was favored, it was evident that students were most concerned with ease of use and efficiency. Only in one instance did a student indicate that it helped him to "see" or understand what he was doing. This observation indicates that possibly students do not perceive concrete objects as being more conceptually engaging.

Conclusion and Summary

At the end of Phase 1, students were able to calculate with reasonable accuracy the subtraction problem using the representation that they had just learned. Conceptual understanding of how and why they borrowed was also reasonable. Transfer of this understanding to the new representation was greater for MN students. With manipulatives after Phase 1, the AL group was able to demonstrate conceptual understanding of the need to borrow, but was unable to convert this to actions when using concrete objects.

This difference in transfer must be interpreted in light of the following points. First, the amount of instructional time required by the MN group to reach criterion in Phase 1 was significantly greater than the AL group. That is, the students in the MN group received 89.2 minutes (58.1%) more instruction on average than the students in the AL group. Second, the MN group, as part of their instruction adapted from the mathematics series *Explorations* (Addison-Wesley, 1988), was taught to map the concrete representation to a symbolic representation. Therefore, the MN group had had exposure to the symbolic representation, whereas the AL group had no exposure to any concrete representations.

After both groups had received instruction with the second representation, the results indicated that the MN students' performance with the symbolic and concrete representation had not improved from the conclusion of the first phase. In fact, one less student was able to calculate the correct answer using the symbolic problem. In contrast, the AL group maintained their level of performance with the symbolic representation, were noticeably more proficient procedurally with the concrete representation, and displayed a greater degree of conceptual understanding.

This difference in procedural accuracy observed in the interviews between the MN and AL groups, however, was not evident in the Posttest and Maintenance Test, where students from both groups performed at a similar level. The accuracy level was only moderate, with the average number of correct responses for both groups being 68% on the Posttest and 60% on the Maintenance Test. Considering the extensive instruction, the scores were surprisingly low.

This study suggests that students can be taught to become somewhat procedurally proficient with either representation, and that conceptual understanding can be developed to a similar level with both treatments. That is, no matter what type of representation is used, if a skill is explicitly taught in a meaningful way, conceptual understanding can be promoted using that representation. This result supports, in part, Resnick and Omanson's (1987) argument that if a symbolic or any other representation is used meaningfully, it can achieve the same aims of developing conceptual knowledge as is (theoretically) claimed for using concrete representations. However, the symbolic and concrete representations are not equally efficient. Instruction takes significantly more time when initial instruction involves a concrete representation. ◆

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Siegfried Engelmann and Douglas Carnine

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